An introduction to continued fractions

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Continued fractions (1)

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots = \sum_{j=0}^{\infty} \frac{1}{j!} \quad (1)$$

$$e = 2 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \ldots}}} \quad (2)$$

$$= 2 + \operatorname*{K}_{j=1}^{\infty} \frac{j}{j} \quad (3)$$

$$= 2 + \frac{2}{2} \, {}_{+} \, \frac{3}{3} \, {}_{+} \, \frac{4}{4} + \cdots \quad (4)$$

Continued fractions (2)

Definition 1. *A* continued fraction *is an expression*

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}} \tag{5}$$

with all $a_j \neq 0$*. The* a_j *and* b_j *are called the* partial numerators *and the* partial denominators *respectively. The continued fraction (5) is denoted as*

$$b_0 + \operatorname*{K}_{j=1}^{\infty} \frac{a_j}{b_j} \text{ or } b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{6}$$

Approximants

Definition 2. *The approximants* f_n *of a continued fraction*

$$f = b_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j}{b_j} \qquad f = b_0 + \frac{a_1}{b_1} \, {}_{+} \, \frac{a_2}{b_2} \, {}_{+} \, \frac{a_3}{b_3} \, {}_{+} \, \cdots \tag{7}$$

are defined as follows

$$f_n = b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_j}{b_j} \qquad f_n = b_0 + \frac{a_1}{b_1} \, {}_{+} \, \cdots \, {}_{+} \, \frac{a_n}{b_n} \tag{8}$$

Examples

$$f = 2 + \frac{2}{2\,+}\,\frac{3}{3\,+}\,\frac{4}{4\,+}\cdots \qquad f = e \tag{9}$$

$$f_0 = 2 \qquad f_0 = 2 \tag{10}$$

$$f_1 = 2 + \frac{2}{2} \qquad f_1 = 3 \tag{11}$$

$$f_2 = 2 + \frac{2}{2\,+}\,\frac{3}{3} \qquad f_2 = \frac{8}{3} = 2.666\ldots \tag{12}$$

$$f_3 = 2 + \frac{2}{2\,+}\,\frac{3}{3\,+}\,\frac{4}{4} \qquad f_3 = \frac{30}{11} = 2.718\ldots \tag{13}$$

Recurrence relations

Let $f = b_0 + \mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$, and define

$$P_{-1} = 1 \qquad Q_{-1} = 0 \tag{14}$$

$$P_0 = b_0 \qquad Q_0 = 1 \tag{15}$$

$$P_n = b_n P_{n-1} + a_n P_{n-2} \qquad Q_n = b_n Q_{n-1} + a_n Q_{n-2} \tag{16}$$

for $n \geq 1$. The following holds:

$$f_n = \frac{P_n}{Q_n} \tag{17}$$

Equivalent continued fractions

Definition 3. *Two continued fractions f and g are called* equivalent *if and only if $f_j = g_j$ for all $j \in \mathbb{N}$.*

Example

$$f = 2 + \frac{2}{2} {}_+ \frac{3}{3} {}_+ \frac{4}{4} {}_+ \frac{5}{5} {}_+ \cdots \qquad = e \tag{18}$$

$$g = 2 + \frac{2}{2} {}_+ \frac{9\cdot 3}{9\cdot 3} {}_+ \frac{9\cdot 4}{4} {}_+ \frac{5}{5} {}_+ \cdots \qquad = e \tag{19}$$

f and g are equivalent continued fractions.

Equivalence transformations (1)

If $f = b_0 + \mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$ is a continued fraction, and $p_j \neq 0$ for all $j \geq 1$, then the continued fraction

$$b_0 + \frac{p_1 a_1}{p_1 b_1} + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{p_{j-1} p_j a_j}{p_j b_j} \tag{20}$$

is equivalent to f.

Equivalence transformations (2)

If $b_j \neq 0$ for all j, the continued fractions

$$b_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j}{b_j} \tag{21}$$

and

$$b_0 + \frac{b_1^{-1} a_1}{1} + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{b_{j-1}^{-1} b_j^{-1} a_j}{1} \tag{22}$$

are equivalent.

Proof. Use expression (20) with $p_j = b_j^{-1}$. □

Examples

$$e = 2 + \frac{2}{2\,+}\,\frac{3}{3\,+}\,\frac{4}{4\,+}\cdots \tag{23}$$

$$= 2 + \frac{1}{1\,+}\,\frac{\frac{1}{2}}{1\,+}\,\frac{\frac{1}{3}}{1\,+}\cdots \tag{24}$$

$$\log 2 = 0 + \frac{1}{1\,+}\,\frac{1^2}{2\,+}\,\frac{1^2}{3\,+}\,\frac{2^2}{4\,+}\,\frac{2^2}{5\,+}\,\frac{3^2}{6\,+}\cdots \tag{25}$$

$$= 0 + \frac{1}{1\,+}\,\frac{\frac{1}{2}}{1\,+}\,\frac{\frac{1}{6}}{1\,+}\,\frac{\frac{1}{3}}{1\,+}\,\frac{\frac{3}{10}}{1\,+}\cdots \tag{26}$$

Tails of series (1)

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \tag{27}$$

$$= \sum_{j=0}^{\infty} \frac{1}{j!} \tag{28}$$

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} \frac{1}{j!} = 0 \tag{29}$$

Tails of series (2)

$$\log 2 = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{30}$$

$$= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} \tag{31}$$

$$\lim_{n\to\infty} \sum_{j=n}^{\infty} \frac{(-1)^{j+1}}{j} = 0 \tag{32}$$

Tails of continued fractions (1)

$$e = 2 + \frac{2}{2} {}_{+} \frac{3}{3} {}_{+} \frac{4}{4} {}_{+} \frac{5}{5} {}_{+} \cdots \tag{33}$$

$$= 2 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{j}{j} \tag{34}$$

$$\lim_{n\to\infty} \mathop{\mathrm{K}}_{j=n}^{\infty} \frac{j}{j} \neq 0 \tag{35}$$

(This can be proven by reductio ad absurdum)

Tails of continued fractions (2)

$$\log 2 = 0 + \frac{1}{1} \underset{+}{} \frac{1}{2} \underset{+}{} \frac{1}{3} \underset{+}{} \frac{4}{4} \underset{+}{} \cdots \tag{36}$$

$$= 0 + \frac{1}{1} \underset{+}{} \operatorname*{K}_{j=2}^{\infty} \frac{\lfloor \frac{j}{2} \rfloor^2}{j} \tag{37}$$

$$\lim_{n\to\infty} \operatorname*{K}_{j=n}^{\infty} \frac{\lfloor \frac{j}{2} \rfloor^2}{j} = \infty \neq 0 \tag{38}$$

Tails of continued fractions (3)

$$f = b_0 + \operatorname*{K}_{j=1}^{\infty} \frac{a_j}{b_j} \qquad f = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots \tag{39}$$

$$f^{(m)} = \operatorname*{K}_{j=m+1}^{\infty} \frac{a_j}{b_j} \qquad f^{(m)} = \frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \cdots \tag{40}$$

$$f_n^{(m)} = \operatorname*{K}_{j=m+1}^{m+n} \frac{a_j}{b_j} \qquad f_n^{(m)} = \frac{a_{m+1}}{b_{m+1}} + \cdots + \frac{a_{m+n}}{b_{m+n}} \tag{41}$$

Modified approximants

$$f = b_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j}{b_j} \qquad\qquad f = b_0 + \frac{a_1}{b_1} \genfrac{}{}{0pt}{}{}{+} \frac{a_2}{b_2} \genfrac{}{}{0pt}{}{}{+} \frac{a_3}{b_3} + \cdots \tag{42}$$

$$S_n(w) = b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_j}{b_j} \genfrac{}{}{0pt}{}{}{+} \frac{w}{1} \qquad S_n(w) = b_0 + \frac{a_1}{b_1} + \cdots + \frac{a_n}{b_n} \genfrac{}{}{0pt}{}{}{+} \frac{w}{1} \tag{43}$$

Note that $S_n(0) = f_n$ and $S_n(f^{(n)}) = f$.

From continued fraction to series (1)

$$f = 2 \qquad + \frac{2}{2} \qquad \genfrac{}{}{0pt}{}{}{+} \frac{3}{3} \qquad \genfrac{}{}{0pt}{}{}{+} \frac{4}{4} \qquad \genfrac{}{}{0pt}{}{}{+ \cdots} \tag{44}$$

$$f_0 = 2 \qquad f_1 = 3 \qquad f_2 = \frac{8}{3} \qquad f_3 = \frac{30}{11} \tag{45}$$

$$f = 2 \qquad + 1 \qquad - \frac{1}{3} \qquad + \frac{2}{33} \qquad + \cdots \tag{46}$$

From continued fraction to series (2)

Let $f = b_0 + \mathrm{K}_{j=0}^{\infty} \frac{a_j}{b_j}$ be a continued fraction, and let Q_i be defined as in (14-16). The following holds:

$$f_n = b_0 + \sum_{j=1}^{n} (-1)^{j+1} \frac{a_1 \cdots a_j}{Q_{j-1} Q_j} \tag{47}$$

Proof. This can be proven using induction and (14-16). □

Definition 4. *We call*

$$b_0 + \sum_{j=1}^{\infty} (-1)^{j+1} \frac{a_1 \cdots a_j}{Q_{j-1} Q_j} \tag{48}$$

the Euler-Minding series *associated to* $f = b_0 + \mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$.

From series to continued fraction (1)

Given a series $\sum_{j=0}^{\infty} c_j$, find a continued fraction $\mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$ such that

$$b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_j}{b_j} = \sum_{j=0}^{n} c_j \tag{49}$$

To obtain this, we define the seqence $(C_j)_j$ of the series' approximants:

$$C_n = \sum_{j=0}^{n} c_j \tag{50}$$

From series to continued fraction (2)

Let

$$\begin{aligned} && b_0 &= C_0 && (51)\\ a_1 &= C_1 - C_0 & b_1 &= 1 && (52)\\ a_j &= \frac{C_{j-1} - C_j}{C_{j-1} - C_{j-2}} & b_j &= \frac{C_j - C_{j-2}}{C_{j-1} - C_{j-2}} && (53) \end{aligned}$$

for $j \geq 2$. Now the following holds:

$$b_0 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{a_j}{b_j} = C_n \tag{54}$$

Example

$$\frac{\pi}{4} = 1 \quad - \frac{1}{3} \quad + \frac{1}{5} \quad - \frac{1}{7} \quad + \cdots \tag{55}$$

$$C_0 = 1 \quad C_1 = \frac{2}{3} \quad C_2 = \frac{13}{15} \quad C_3 = \frac{76}{105} \tag{56}$$

$$f = 1 \quad - \frac{\frac{1}{3}}{1} \quad + \frac{\frac{3}{5}}{\frac{2}{5}} \quad + \frac{\frac{5}{7}}{\frac{2}{7}} \quad + \cdots \tag{57}$$

Successive substitution

How to create a continued fraction expansion for a value/function f?

$$f = b_0 + f^{(0)} \tag{58}$$

$$f^{(0)} = \frac{a_1}{b_1 + f^{(1)}} \tag{59}$$

$$f^{(1)} = \frac{a_2}{b_2 + f^{(2)}} \tag{60}$$

$$\cdots \tag{61}$$

Example

$$\sqrt{2} = 1 + (\sqrt{2} - 1) \qquad f^{(0)} = \sqrt{2} - 1 \tag{62}$$

$$\sqrt{2} - 1 = \frac{1}{2 + (\sqrt{2} - 1)} \qquad f^{(1)} = \sqrt{2} - 1 \tag{63}$$

$$\sqrt{2} = \lim_{n \to \infty} \left(1 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{1}{2} \right) \tag{64}$$

$$= 1 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{1}{2} \tag{65}$$

Warning!

$$-\sqrt{2} = 1 + (-\sqrt{2} - 1) \qquad f^{(0)} = -\sqrt{2} - 1 \tag{66}$$

$$-\sqrt{2} - 1 = \frac{1}{2 + (-\sqrt{2} - 1)} \qquad f^{(1)} = \sqrt{2} - 1 \tag{67}$$

$$-\sqrt{2} \neq \lim_{n\to\infty} \left(1 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{1}{2} \right) \tag{68}$$

but if we modify the approximants

$$-\sqrt{2} = \lim_{n\to\infty} \left(1 + \mathop{\mathrm{K}}_{j=1}^{n} \frac{1}{2} + \frac{-1 - \sqrt{2}}{1} \right) \tag{69}$$

Exercises

1. Apply an equivalence transformation to the following continued fraction expansion of $\sqrt{2}$

$$\sqrt{2} = 1 + \frac{1}{2} {}_{+} \frac{1}{2} {}_{+} \cdots \tag{70}$$

   such that all partial denominators of the transformed continued fraction equal 1.

2. Calculate (the first terms of) the Euler-Minding series of the continued fraction (70).

Correspondence

Definition 5. *A continued fraction*

$$f = b_0(x) + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j(x)}{b_j(x)} \tag{71}$$

is said to be corresponding *to a power series* $\sum_{j=0}^{\infty} c_j x^j$ *if for each* $n \geq 0$

$$\sum_{j=0}^{n} c_j x^j \tag{72}$$

matches the first $n+1$ *terms of the Taylor expansion of* f_n.

Method of Viscovatov

For a power series $\sum_{j=0}^{\infty} c_j x^j$, with $c_j \neq 0$ for all $j \geq 0$, define

$$d_{00} = 1 \tag{73}$$

$$d_{0k} = 0 \qquad (k > 1) \tag{74}$$

$$d_{1k} = c_k \qquad (k \geq 0) \tag{75}$$

$$d_{jk} = d_{j-1,0} d_{j-2,k+1} - d_{j-2,0} d_{j-1,k+1} \qquad (j \geq 2, k \geq 0) \tag{76}$$

The following holds:

$$\lim_{n\to\infty} \sum_{j=0}^{n} c_j x^j = \lim_{n\to\infty} \frac{d_{10}}{1} + \operatorname*{K}_{j=1}^{n} \frac{d_{j+1,0}\, x}{d_{j0}} \tag{77}$$

Constructing corresponding continued fractions using Viscovatov

$$f(x) = \sum_{j=0}^{\infty} c_j x^j \qquad c_1 \neq 0 \tag{78}$$

$$\frac{f(x) - c_0}{x} = \sum_{j=0}^{\infty} c_{j+1} x^j \tag{79}$$

$$= \frac{d_{10}}{1} + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{d_{j+1,0}\, x}{d_{j0}} \qquad \text{(Viscovatov)} \tag{80}$$

$$f(x) = c_0 + \frac{d_{10} x}{1} + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{d_{j0}\, x}{d_{j-1,0}} \tag{81}$$

The continued fraction (81) corresponds to the series (78).

Example (1)

$$\log(1+x) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{82}$$

$$\frac{\log(1+x) - 0}{x} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots \tag{83}$$

$$(d_{jk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -\frac{1}{2} & \frac{1}{3} & -\frac{1}{4} & \ldots\ldots & \\ \frac{1}{2} & -\frac{1}{3} & \frac{1}{4} & \ldots\ldots\ldots\ldots & & \\ \frac{1}{12} & -\frac{1}{12} & \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ \frac{1}{18} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & & \end{pmatrix} \tag{84}$$

Example (2)

We get

$$\frac{\log(1+x)}{x} = \frac{1}{1} + \frac{\frac{1}{2}x}{1} + \frac{\frac{1}{12}x}{\frac{1}{2}} + \frac{\frac{1}{18}x}{\frac{1}{12}} + \cdots \tag{85}$$

which leads to

$$\log(1+x) = 0 + \frac{x}{1} + \frac{\frac{1}{2}x}{1} + \frac{\frac{1}{12}x}{\frac{1}{2}} + \frac{\frac{1}{18}x}{\frac{1}{12}} + \cdots \tag{86}$$

Example (3)

n	f_n	Taylor expansion
1	$\frac{x}{1}$	$0 + x$
2	$\frac{x}{1} + \frac{\frac{1}{2}x}{1}$	$0 + x - \frac{1}{2}x^2 + \frac{1}{4}x^3 - \cdots$
3	$\frac{x}{1} + \frac{\frac{1}{2}x}{1} + \frac{\frac{1}{12}x}{\frac{1}{2}}$	$0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{2}{9}x^4 + \cdots$

Approximant evaluation

How to calculate the value of an (unmodified) approximant f_n of $f = \mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$?

Backward evaluation. Define $r_n = 0$, and $r_{j-1} = \frac{a_j}{b_j + r_j}$ for $j = 0, \ldots, n-1$. Now $f_n = b_0 + r_0$.

Forward evaluation. Remember e.g. the recurrence relations (14-16).

A tridiagonal system

Theorem. *The* n*th approximant of the continued fraction* $\mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$ *is the first unknown* $x_{1,n}$ *of the tridiagonal system*

$$\begin{pmatrix} b_1 & -1 & 0 & \dots & 0 \\ a_2 & b_2 & -1 & & \vdots \\ 0 & a_3 & b_3 & \ddots & 0 \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \dots & 0 & a_n & b_n \end{pmatrix} \begin{pmatrix} x_{1,n} \\ \vdots \\ x_{n,n} \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad (87)$$

Illustration

Use Gaussian elimination to transform the system (87) into a lower triangular matrix:

$$\begin{pmatrix} b_1 & -1 & 0 \\ a_2 & b_2 & -1 \\ 0 & a_3 & b_3 \end{pmatrix} \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ x_{3,n} \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} \tag{88}$$

$$\begin{pmatrix} b_1 + \frac{a2}{b_2 + \frac{a3}{b_3}} & 0 & 0 \\ a_2 & b_2 + \frac{a_3}{b_3} & 0 \\ 0 & a_3 & b_3 \end{pmatrix} \begin{pmatrix} x_{1,n} \\ x_{2,n} \\ x_{3,n} \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ 0 \end{pmatrix} \tag{89}$$

You can prove the theorem in a similar way using induction.

The other way round

If you use Gaussian elimination to convert the system into an upper triangular matrix, you will find after backsubstitution:

$$x_{1,n} = \sum_{j=1}^{n} (-1)^{j-1} \frac{a_1 \cdots a_j}{h_1^2 \cdots h_{j-1}^2 h_j} \tag{90}$$

with

$$h_1 = b_1 \text{ and } h_j = b_j + \frac{a_j}{h_{j-1}} \text{ for } j \geq 2 \tag{91}$$

This leads to the following forward algorithm:

$$x_{1,n} = x_{1,n-1} - (-1)^n \frac{a_1 \cdots a_n}{h_1^2 \cdots h_{n-1}^2 h_n} \tag{92}$$

Link with Euler-Minding

Expression (90) is closely related to the Euler-Minding series of the continued fraction (48). Sketch of the proof:

- Prove for the sequence $(h_n)_n$ as defined in (91) that $h_j = \frac{Q_j}{Q_{j-1}}$ for all $j > 0$. ($(Q_n)_n$ is defined in (14–16)).
- Then $Q_{j-1}Q_j = h_1^2 \cdots h_{j-1}^2 h_j$

Function evaluation using continued fractions

The evaluation of $F(z)$ generally takes three steps

$$z \xrightarrow{A} z' \xrightarrow{G} y \xrightarrow{P} F(z) \qquad (93)$$

A argument reduction

G a function we can easily calculate using a continued fraction expansion; usually $G = F$

P ‘post processing’ (depends on both z and y)

Truncation error

Since we won't be able to calculate

$$G(x) = b_0 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{a_j(x)}{b_j(x)} \tag{94}$$

we will approximate $G(x)$ by a modified n'th approximant

$$S_n(w;x) = b_0 + \frac{a_1(x)}{b_1(x)} + \frac{a_2(x)}{b_2(x)} + \cdots + \frac{a_n(x)}{b_n(x)} + \frac{w}{1} \tag{95}$$

The choice of n can be made

- a priori
- a posteriori

Error bounding strategy

Approximating $G(x)$ by $S_n(w;x)$ introduces an error ε

$$S_n(w;x) = G(x) + \varepsilon \tag{96}$$

We want to make sure that $|\varepsilon|$ is smaller than some upper bound, $\overline{\varepsilon}$ (e.g. $\overline{\varepsilon} = 2^{-52}$). Suppose we can bound ε by some expression E which depends on parameters $p_1, p_2, \ldots$

$$|\varepsilon| \leq E(p_1, p_2, \ldots) \tag{97}$$

If we choose our parameters $p_1, p_2, \ldots$ such that

$$E(p_1, p_2, \ldots) \leq \overline{\varepsilon} \tag{98}$$

then indeed

$$|\varepsilon| \leq \overline{\varepsilon} \tag{99}$$

Henrici-Pfluger (HP)

Suppose $f = b_0 + \mathrm{K}_{j=1}^{\infty} \frac{a_j}{1}$ is a converging continued fraction with all $a_n > 0$. Then for all $n \geq 1$ the following holds:

$$|f - f_n| \leq |f_n - f_{n-1}| \tag{100}$$

Determining n a posteriori using HP

$$f = \frac{1}{1} + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{\frac{(j-1)^2}{4(j-1)^2-1}}{1} \rightarrow \frac{\pi}{4} = 0.78539816\ldots \qquad (101)$$

n	f_n	$\lvert f_n - \frac{\pi}{4}\rvert$	$\lvert f_n - f_{n-1}\rvert$
1	1	$0.21460\ldots$	1
2	0.75	$0.035398\ldots$	0.25
3	$0.79166\ldots$	$0.0062685\ldots$	$0.041666\ldots$
4	$0.78431\ldots$	$0.0010844\ldots$	$0.0073529\ldots$
5	$0.7855855\ldots$	$0.00018742\ldots$	$0.00127186\ldots$
6	$0.785368536\ldots$	$3.23097\ldots \cdot 10^{-5}$	$0.00021973\ldots$

Interval Sequence Theorem (IST)

Suppose $f = \mathrm{K}_{j=1}^{\infty} \frac{a_j}{1}$. If we can find sequences $(\ell_n)_n$ and $(r_n)_n$ such that for all n

1. $0 < \ell_n < r_n < \infty$
2. $(1 + r_n)\ell_{n-1} \leq a_n \leq (1 + \ell_n)r_{n-1}$

then we can apply the 'interval sequence theorem':

$$|f - S_n(w)| \leq (r_n - \ell_n)\frac{r_0}{1 + \ell_n}\prod_{k=1}^{n-1}\frac{r_k}{1 + r_k} \tag{102}$$

for $w \in [\ell_n, r_n]$.

Sufficient conditions for the IST

In general, we can find suitable ℓ_n and r_n if

- the partial numerators are non-decreasing towards a positive number.
- the partial numerators are non-increasing towards zero.
- the even partial numerators are non-decreasing towards a positive number a, and the odd partial numerators are non-increasing towards a positive number b such that $a \leq b$.
- the partial numerators are non-decreasing towards zero.
- the partial numerators are non-decreasing towards infinity.

Example

$$f = \frac{1}{1} + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{\frac{(j-1)^2}{4(j-1)^2-1}}{1} \rightarrow \frac{\pi}{4} = 0.78539816\ldots \qquad (103)$$

n	$S_n(w)$	$\|f - S_n(w)\|$	expr. (102)
1	$0.80599\ldots$	$0.020592\ldots$	$0.049945\ldots$
2	$0.78451\ldots$	$0.00087954\ldots$	$0.0022574\ldots$
3	$0.78546\ldots$	$6.7121\ldots \cdot 10^{-5}$	$0.00017928\ldots$
4	$0.78539\ldots$	$6.4734\ldots \cdot 10^{-6}$	$1.7757\ldots \cdot 10^{-5}$
5	$0.7853988\ldots$	$7.1024\ldots \cdot 10^{-7}$	$1.9847\ldots \cdot 10^{-6}$
6	$0.7853980\ldots$	$8.4557\ldots \cdot 10^{-8}$	$2.3953\ldots \cdot 10^{-7}$

Determining n a priori using IST (1)

If $\left(\frac{r_n-\ell_n}{1+\ell_n}\right)_n$ and $\left(\frac{r_n}{1+r_n}\right)_n$ are decreasing sequences, we can write for $\varepsilon = |f - S_n(w)|$:

$$\varepsilon \leq (r_n - \ell_n)\frac{r_0}{1+\ell_n}\prod_{k=1}^{n-1}\frac{r_k}{1+r_k} \tag{104}$$

$$< (r_H - \ell_H)\frac{r_0}{1+\ell_H}\left(\prod_{k=1}^{H}\frac{r_k}{1+r_k}\right)\left(\frac{r_H}{1+r_H}\right)^{n-H} \quad \text{for } H < n \tag{105}$$

$$= (r_H - \ell_H)\frac{r_0}{1+\ell_H}\left(\prod_{k=1}^{H}\frac{1}{1+\frac{1}{r_k}}\right)\left(\frac{1}{1+\frac{1}{r_H}}\right)^{n-H} \tag{106}$$

Determining n a priori using IST (2)

To bound the error by $\overline{\varepsilon}$ (e.g. $\overline{\varepsilon} \leq 2^{-52}$), it suffices to bound this expression:

$$(r_H - \ell_H)\frac{r_0}{1+\ell_H}\left(\prod_{k=1}^{H}\frac{1}{1+\frac{1}{r_k}}\right)\left(\frac{1}{1+\frac{1}{r_H}}\right)^{n-H} < \overline{\varepsilon} \qquad (107)$$

It follows that

$$n > \frac{1}{\log\left(1+\frac{1}{r_H}\right)}\Bigg[\log(r_H - \ell_H) + \log r_0 - \log(1+\ell_H) - \left(\sum_{k=1}^{H}\log\left(1+\frac{1}{r_k}\right)\right) - \log\overline{\varepsilon} + H\Bigg] \qquad (108)$$

Determining n a priori using IST (3)

$$f = \frac{1}{1} + \mathop{\mathrm{K}}_{j=2}^{\infty} \frac{\frac{(j-1)^2}{4(j-1)^2-1}}{1} \to \frac{\pi}{4} = 0.78539816\ldots \tag{109}$$

error bound	real n	est. n
10^{-1}	1	1
10^{-2}	2	2
10^{-3}	2	3
10^{-4}	3	4
10^{-5}	4	5
10^{-6}	5	6

Excercises

1. If $|z| < 1$, then $\arctan(z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{2j+1}$ is a power series expansion of $\arctan(z)$. Calculate the first terms of a corresponding continued fraction using Viscovatov's method.

2. If a continued fraction g is constructed from a continued fraction f using the equivalence transformation (20), what is the connection between the tails $f^{(n)}$ and $g^{(n)}$?

3. What is the limit of the tails of the continued fraction (25)? **Hint:** The limit of the tails of (26) equals $\frac{\sqrt{2}-1}{2}$.

The problem $1 + \mathrm{K}_{j=1}^{\infty} \frac{1}{2}$

Evaluate

$$1 + \underset{j=1}{\overset{\infty}{\mathrm{K}}} \frac{1}{2} + \frac{w}{1} \tag{110}$$

for different values of w

n	1	2	3	4
$w = 0$	1.5	1.4	1.417	1.414
$w = 1$	1.333	1.429	1.412	1.415
$w = -1$	2	1.333	1.429	1.412
$w = -1 + \sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
$w = -1 - \sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$	$-\sqrt{2}$

Periodic continued fractions

Definition 6. *A continued fraction* $f = b_0 + \mathrm{K}_{j=1}^{\infty} \frac{a_j}{b_j}$ *is called* 1-periodic *if* $a_n = A$ *and* $b_n = B$ *for all* $n \geq 1$*.*

If $n \geq 1$, then a modified n'th approximant of f satisifies

$$b_0 + \operatorname*{K}_{j=0}^{n} \frac{a_j}{b_j} + \frac{w}{1} = b_0 + T^n(w) \qquad (111)$$

where

$$T(w) = \frac{A}{B+w} \text{ and } T^n(w) = \underbrace{T \circ T \circ \cdots \circ T}_{n}(w) \qquad (112)$$

Linear fractional transformations (LFT)

Definition 7. *A* linear fractional transformation *is a real function of the form*

$$T(w) = \frac{aw + b}{cw + d} \tag{113}$$

with $ad - bc \neq 0$.

If we know what happens to $T \circ T \circ \cdots \circ T(w)$, we might be able to tell more about our case (111), where

$$T(w) = \frac{A}{B + w} \tag{114}$$

(i.e. $a = 0$, $b = A$, $c = 1$, $d = B$)

Iterations of LFT's (1)

Suppose T is an arbitrary LFT, $w \in \mathbb{R}$, and suppose a real number x exists such that

$$\lim_{n\to\infty} T^n(w) = x \tag{115}$$

Expression (115) implies that $T(x) = x$, because

$$\lim_{n\to\infty} T^n(w) = x \tag{116}$$

$$T\left(\lim_{n\to\infty} T^n(w)\right) = x \tag{117}$$

$$T(x) = x \tag{118}$$

If T is not the identity function, there are at most 2 fixed points x such that $T(x) = x$.

Iterations of LFT's (2)

If $T = \frac{aw+b}{cw+d}$ is a LFT with (complex) fixed points[a] x and y.

- If $x = y$ then $(T^n(w))_n$ converges and

$$\lim_{n\to\infty} T^n(w) = x \qquad \text{for all } w \in \mathbb{C} \tag{119}$$

- If $x \neq y$ and

$$|cx + d| = |cy + d| \qquad \text{if } c \neq 0 \tag{120}$$

$$|a| = |d| \qquad \text{if } c = 0 \tag{121}$$

then $(T^n(w))_n$ diverges for all $w \in \mathbb{C} \setminus \{x, y\}$.

[a] we allow ∞ as fixed point

Iterations of LFT's (3)

- If $x \neq y$ and

$$|cx + d| > |cy + d| \qquad \text{if } c \neq 0 \qquad (122)$$

$$|a| \neq |d| \qquad \text{if } c = 0 \qquad (123)$$

then $(T^n(w))_n$ converges and

$$\lim_{n\to\infty} T^n(w) = x \qquad \text{for all } w \in \mathbb{C} \setminus \{y\} \qquad (124)$$

Example

For the continued fraction

$$f = 1 + \mathop{\mathrm{K}}_{j=1}^{\infty} \frac{1}{2} \tag{125}$$

we have that

$$T = \frac{1}{2 + w} \tag{126}$$

which has fixed points $x = -1 + \sqrt{2}$ and $y = -1 - \sqrt{2}$. Because $a = 0, b = 1, c = 1, d = 2$, we are in case (122).

Example (continued)

$$\lim_{n\to\infty} \operatorname*{K}_{j=1}^{n} \frac{1}{2} + \frac{w}{1} = \lim_{n\to\infty} T^n(w) \tag{127}$$

$$= -1 + \sqrt{2} \qquad \text{for } w \neq -1 - \sqrt{2} \tag{128}$$

$$1 + \lim_{n\to\infty} \operatorname*{K}_{j=1}^{n} \frac{1}{2} + \frac{w}{1} = \sqrt{2} \qquad \text{for } w \neq -1 - \sqrt{2} \tag{129}$$