Blending interval and set arithmetic for maximal reliability

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Abstract.
In both [3] and [8], the authors review the implementation of the basic operations in interval arithmetic, and in particular discuss the different approaches given in the literature for interval division when the divisor interval contains zero. In these papers, and in the references therein, the basic operations are defined for real or extended real interval operands.
Division by an interval containing zero is a special case of an interval function for which the input arguments contain points outside the domain of the underlying point function. A number of approaches exist in the literature [7, 12] to remove restrictions on the domain of interval functions and hence obtain a closed, exception-free interval system.
In this paper we present an alternative approach to remove restrictions on the domain of interval functions and to guarantee the inclusion property in all situations, even when some input intervals contain points that lie outside the domain of the underlying point function. To achieve this, we allow for the (efficient) set-based representation of non-real results. The computed intervals are sharp, yet contain more information and the resulting interval system is closed and exception-free. We also show how the presented ideas can be implemented in an interval arithmetic library. The performance overhead is negligible compared to the fact that the implementation using the new approach offers 100% reliability in return.
The structure of the paper is as follows. We set off with a motivating example in *Section* 1. In *Section* 2 we review various approaches to interval division and then introduce vset-division of real intervals, based on the newly introduced concept of value set or vset. In *Section* 3 we give a formal definition of real vset-intervals and arithmetic on these intervals. We prove a number of essential properties and point out the likenesses and differences with other approaches. Finally, in *Section* 4, we discuss the implementation of vset-interval arithmetic in a floating-point context.

1 Motivation.

It is well-known that without the proper handling of special cases, real floating-point arithmetic and real floating-point interval arithmetic systems are not closed. Besides mathematically undefined results, complex results can not be represented nor approximated in such systems. Programming environments such as MATLAB and computer algebra systems, both of which are often used for prototyping, automatically switch from real to complex mode when the mathematical result is complex. This is not an option in typed programming languages and special values such as NaN, introduced in the IEEE 754 standard for floating-point arithmetic [1], are needed to obtain a closed floating-point system.
To obtain a closed system for interval arithmetic, several approaches have been proposed in the literature [3, 8, 12]. These approaches compute comparable interval bounds for real results but in the case of non-real results they often do not reflect all that is known about the interval-valued expressions. Consider evaluating the function

$$g(x) = \sqrt{1 - \left(\frac{\cos x}{x - \pi}\right)^2}$$

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over the interval $[3.14, 3.15]$. Mathematically, we have that

$$x \in \mathbb{R} : \left(\frac{\cos x}{x - \pi}\right)^2 > 1 \Longrightarrow g(x) \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$
$$g(\pi) = \infty$$

and so $[3.14, 3.15]$ is not in the domain of g, when considered as a function from $\mathbb{R}$ to $\mathbb{R}$. As will become clear, how well $g([3.14, 3.15])$ can be computed/represented, is influenced by the programming environment in which it is evaluated. In the most expressive programming environments, such as computer algebra systems, we expect the full range of the function g over the interval to be returned. We indeed find with Mathematica v4.0

$$\begin{aligned} g([3.14, 3.15]) &= \sqrt{1 - \left(\frac{\cos [3.14, 3.15]}{[3.14, 3.15] - \pi}\right)^2} \\ &= \sqrt{1 - \left(\frac{[-1.00, -9.99 \times 10^{-1}]}{[-0.002, 0.008]}\right)^2} \\ &= \sqrt{1 - ([-\infty, -118] \cup [627, +\infty])^2} \\ &= \sqrt{1 - [14146, +\infty]} \\ &= \sqrt{[-\infty, -14145]} \end{aligned}$$

On the other hand, in the interval library INTLAB [9] for MATLAB, which is based on IEEE double precision floating-point arithmetic, the result returned is

$$g([3.14, 3.15]) = \mathrm{NaN} + \mathrm{NaN}\ \mathrm{i}$$

While this result is not informative, it is consistent with the underlying IEEE arithmetic. Indeed, computing the range of g over the interval $[3.14, 3.15]$ comes down to taking the square root of (finite and infinite) negative arguments. For arguments $x < 0$, IEEE floating-point arithmetic, on which MATLAB/INTLAB is based, states that $\sqrt{x} = \mathrm{NaN}$.

If we use classical typed programming languages with real data types, we cannot expect the result to be as expressive as in computer algebra systems. Using the SUN Forte compilers [10] which have interval arithmetic built-in, we find

$$\begin{aligned} g([3.14, 3.15]) &= \sqrt{1 - \left(\frac{\cos [3.14, 3.15]}{[3.14, 3.15] - \pi}\right)^2} \\ &= \sqrt{1 - \left(\frac{[-1.00, -9.99 \times 10^{-1}]}{[-0.002, 0.008]}\right)^2} \\ &= \sqrt{1 - [-\infty, +\infty]^2} \\ &= \sqrt{1 - [0, +\infty]} \\ &= \sqrt{[-\infty, 1]} \\ &= [0, 1] \end{aligned}$$

The interval $[0, 1]$ which is returned, is the result of an overestimation of $g([3.14, 3.15]) \cap \mathbb{R} = \emptyset$: for no $x \in [3.14, 3.15]$ does $g(x)$ belong to $[0, 1]$. Furthermore, since no exceptions are raised by the SUN Forte compilers, the user is not aware that the result returned does not reflect the nature of the mathematical result. Such an approach is fine if the user is only interested in real-valued results. The purpose of this paper is to introduce a closed interval system in which the computed intervals are sharp yet contain more information, making our approach mathematically reliable yet implementable in a typed programming language. To achieve this, we allow for an extended interval representation. An interval is represented by its left and right endpoint and an additional parameter, which can be represented with only few bits, to reflect any non-real results. Using this extended representation, we find, when evaluating $g([3.14, 3.15])$:

$$\begin{aligned} g([3.14, 3.15]) &= \sqrt{1 - \left(\frac{\cos[3.14, 3.15]}{[3.14, 3.15] - \pi}\right)^2} \\ &= \sqrt{1 - \left(\frac{[-1.00, -9.99 \times 10^{-1}]}{[-0.002, 0.008]}\right)^2} \\ &= \sqrt{1 - (]-\infty, +\infty[\cup \{\infty\})^2} \\ &= \sqrt{1 - [0, +\infty[\cup \{\infty\}} \\ &= \sqrt{]-\infty, 1] \cup \{\infty\}} \\ &= [0, 1] \cup \{\mathbb{C}_0, \infty\} \qquad \mathbb{C}_0 = \mathbb{C} \setminus \{0\} \end{aligned}$$

This result can be interpreted as follows: the real subset of $g([3.14, 3.15])$ is contained in $[0, 1]$ and there exists at least one $x \in [3.14, 3.15]$ such that $g(x) \in \mathbb{C}_0 \setminus \mathbb{R}$ and at least one $y \in [3.14, 3.15]$ such that $g(y) = \infty$. That is, the true nature of the mathematical result is represented and no exceptions need to be raised. In the next sections, we formalize the arithmetic on this extended interval representation, which we refer to as value set (vset) intervals.

2 Interval division revisited.

Ever since interval arithmetic was introduced by R. Moore in [5], there have been several proposals in the literature on how best to define division of intervals when the divisor contains 0. We refer to [8] and [3] for good overviews. Division by an interval containing 0 is a very simple example of the evaluation of an interval function in a point outside its domain, if we consider point-wise division as a function from $\mathbb{R}^2$ to $\mathbb{R}$. We therefore start our discussion by looking at different existing definitions of interval division.

In the sequel of the text we shall distinguish between the affine extended real line $\overline{\mathbb{R}}_A = \mathbb{R} \cup \{-\infty, +\infty\}$ and the projective extended real line $\overline{\mathbb{R}}_P = \mathbb{R} \cup \{\infty\}$ and also refer to the extended complex plane as $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. With the usual notation

$$\begin{aligned} [a, b] &= \{x \in \overline{\mathbb{R}}_A \mid a \le x \le b\} \\ ]a, b[&= \{x \in \overline{\mathbb{R}}_A \mid a < x < b\} \end{aligned}$$

we shall denote by $\mathbb{I}_f$, the set of finite real intervals

$$\mathbb{I}_f = \{[a, b] \mid a, b \in \mathbb{R}\}$$

by $\mathbb{I}_r$ the set of real intervals

$$\mathbb{I}_r = \mathbb{I}_f \cup \{]-\infty, b] \mid b \in \mathbb{R}\} \cup \{[a, +\infty[\mid a \in \mathbb{R}\} \cup \{]-\infty, +\infty[\} \tag{2.1}$$

and by $\mathbb{I}_e$ the set of extended real intervals

$$\mathbb{I}_e = \{[a, b] \mid a, b \in \overline{\mathbb{R}}_A\}$$

When the distinction between the different sets of intervals is not relevant, we shall simply use the notation $\mathbb{I}$. It is clear that $\mathbb{I}_f$ and $\mathbb{I}_r$ are subsets of $2^{\mathbb{R}}$ while $\mathbb{I}_e \subset 2^{\overline{\mathbb{R}}_A}$.

Division is a special case of a binary function f with domain $D_f \subset \mathbb{R}^2$. From [8] and [3] we recall the two main definitions for interval division. From [12] we repeat the definition based on containment sets. In these definitions, we need the concept of interval hull or hull for short.

Definition 1

Let S be an arbitrary subset of $\mathbb{R}$. The interval hull of S in the set of intervals $\mathbb{I}$ is defined as the smallest element of $\mathbb{I}$ containing S.

It is clear that a similar definition can be given when S is a subset of $\overline{\mathbb{R}}_A$. In that case $\mathbb{I} \equiv \mathbb{I}_e$.

Definition 2 [3]

Let $X, Y \in \mathbb{I}_r$ be real, non-empty intervals. Then we define

$$\begin{aligned} \div_F : \mathbb{I}_r^2 &\longrightarrow 2^{\mathbb{R}} \\ (X,Y) &\longrightarrow X \div_F Y = \{z \in \mathbb{R} \mid \exists x \in X, y \in Y \setminus \{0\}, z = x/y\} \end{aligned} \tag{2.3a}$$

and the corresponding interval division

$$\begin{aligned} /_F : \mathbb{I}_r^2 &\longrightarrow \mathbb{I}_r \\ (X,Y) &\longrightarrow X/_F Y = \mathrm{hull}_{\mathbb{I}_r}(X \div_F Y) \end{aligned} \tag{2.3b}$$

As in [3], we shall refer to this division as functional division. It is clear from (2.3), that when $X, Y \in \mathbb{I}_r$, the set $X \div_F Y$ is not necessarily a real interval. For example, $[1,1] \div_F [-1,1] =]-\infty,-1] \cup [1,+\infty[$. In this case, the result of the interval division is $X/_F Y = \mathrm{hull}(X \div_F Y) =]-\infty,+\infty[\supset X \div_F Y$. A similar remark holds for the subsequent definitions of interval division.
More importantly, one can wonder whether, from a mathematical point of view, it is correct to ignore in $[1,1] \div_F [-1,1]$ the division by 0, mainly because the correct mathematical result, projective ∞, cannot be represented as part of the result interval. And if so, should the user/programmer be notified about this fact? If not, the user/programmer is left unaware of the fact that a function, in this case division, may not be evaluated in all points of the input intervals.
The next definition of interval division avoids the problem of division by 0 in the following way.

Definition 3 [8]

Let $X, Y \in \mathbb{I}_r$ be real, non-empty intervals, then we define

$$\begin{aligned} \div_R : \mathbb{I}_r^2 &\longrightarrow 2^{\mathbb{R}} \\ (X,Y) &\longrightarrow X \div_R Y = \{z \in \mathbb{R} \mid \exists x \in X, y \in Y, z \cdot y = x\} \end{aligned} \tag{2.5a}$$

and the corresponding interval division

$$\begin{aligned} /_R : \mathbb{I}_r^2 &\longrightarrow \mathbb{I}_r \\ (X,Y) &\longrightarrow X/_R Y = \mathrm{hull}_{\mathbb{I}_r}(X \div_R Y) \end{aligned} \tag{2.5b}$$

It was shown in [3] that the following relation holds between functional division and the interval division (2.5), sometimes referred to as relational division.

Lemma 2.1: [3]

Let $X, Y \in \mathbb{I}_r$ be real intervals then $X \div_F Y \subseteq X \div_R Y$. Furthermore,

$$X \div_R Y = \begin{cases} X \div_F Y & 0 \notin X \cap Y \\ ]-\infty,+\infty[& \text{else} \end{cases}$$

and

$$X/_F Y \subseteq X/_R Y$$

While the relational division has been implemented in some libraries for interval arithmetic, we shall not discuss it further in this paper because, unlike *Definition* 2 and *Definition* 5, it does not easily generalize to other interval functions for which the input arguments contain points outside the domain of the underlying point function.
The last definition for interval division we recall here is introduced in [12] and we shall refer to it as the containment set division. We therefore first recall the notion of containment set.

Definition 4 [12]

Let f be a function of n variables and let $\mathbf{x} = (x_1, \dots, x_n)$. The containment set of the function $f : \overline{\mathbb{R}}_A^n \to \overline{\mathbb{R}}_A$ with domain D_f evaluated at the point $\mathbf{x}$ is denoted by $\mathrm{cset}(f, \{\mathbf{x}\})$ and is defined as follows:

$$\mathrm{cset}(f, \{\mathbf{x}\}) = \{z \in \overline{\mathbb{R}}_A \mid \exists (\mathbf{x}_k)_{k\in\mathbb{N}} \subseteq D_f, \lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}, z \text{ is accumulation point of } (f(\mathbf{x}_k))\}$$

We remark that when $\mathbf{x} \in D_f$ and f is continuous in $\{\mathbf{x}\}$, $\mathrm{cset}(f, \{\mathbf{x}\}) = \{f(\mathbf{x})\}$, while when $\mathbf{x} \notin \overline{D}_f$, where $\overline{D}_f$ denotes the closure of the domain D_f, $\mathrm{cset}(f, \{\mathbf{x}\}) = \emptyset$. When $\mathbf{x} \in \overline{D}_f \setminus D_f$, no such general statement can be made about $\mathrm{cset}(f, \{\mathbf{x}\})$. For example, when f equals division we find

$$\begin{aligned} &\mathrm{cset}(/, \{(x, 0)\}) = \{-\infty, +\infty\} && x \neq 0 \in \mathbb{R} \\ &\mathrm{cset}(/, \{(0, 0)\}) = [-\infty, +\infty] = \overline{\mathbb{R}}_A \end{aligned}$$

Based on the concept of containment set, we have the following definition for interval division.

Definition 5 [12]

Let $X, Y \in \mathbb{I}_r$ be real, non-empty intervals, then we define

$$\begin{aligned} \div_C : \mathbb{I}_r^2 &\longrightarrow 2^{\overline{\mathbb{R}}_A} \\ (X, Y) &\longrightarrow X \div_C Y = \mathrm{cset}(/, (X, Y)) = \{z \in \overline{\mathbb{R}}_A \mid \exists x \in X, y \in Y, z \in \mathrm{cset}(/, \{(x, y)\})\} \end{aligned} \tag{2.7a}$$

and the corresponding containment set interval division

$$\begin{aligned} /_C : \mathbb{I}_r^2 &\longrightarrow \mathbb{I}_e \\ (X, Y) &\longrightarrow X/_C Y = \mathrm{hull}_{\mathbb{I}_e}(X \div_C Y) \end{aligned} \tag{2.7b}$$

According to *Definition* 5 we have $[1, 1] \div_C [-1, 1] = [-\infty, -1] \cup [1, +\infty]$, while $[0, 2] \div_C [0, 1] = \overline{\mathbb{R}}_A \supset [0, 2] \div_R [0, 1] = \left]-\infty, +\infty\right[\supset [0, 2] \div_F [0, 1] = [0, +\infty[$. In general, we have the following relation between containment set division and relational division.

Lemma 2.2:

Let $X, Y \in \mathbb{I}_r$ be real intervals then

$$X \div_R Y = (X \div_C Y) \cap \mathbb{R} \equiv \mathrm{cset}(/, (X, Y)) \cap \mathbb{R}$$

and

$$X/_R Y \subseteq (X/_C Y) \cap \mathbb{R}$$

The short proof is given in Appendix A.

We shall now indicate how functional and containment set division can be recovered as special cases of a more general definition. To this end, we introduce the notion of value set or vset.

Definition 6

Let f be a rational function with real coefficients of n variables and let $\mathbf{x} = (x_1, \dots, x_n)$. The value set of the function $f : \mathbb{R}^n \to \overline{\mathbb{R}}_P$ with domain D_f evaluated at the point $\mathbf{x}$ is denoted by $\mathrm{vset}(f, \{\mathbf{x}\})$ and is defined as follows:

$$\mathrm{vset}(f, \{\mathbf{x}\}) = \begin{cases} \{f(\mathbf{x})\} & f(\mathbf{x}) \in \overline{\mathbb{R}}_P \\ \{\mathrm{NaN}\} & \mathbf{x} \notin D_f \end{cases}$$

For any subset $\mathbf{X} \subseteq \mathbb{R}^n$, we define

$$\text{vset}(f, \mathbf{X}) = \bigcup_{\mathbf{x} \in \mathbf{X}} \text{vset}(f, \{\mathbf{x}\})$$

The value set of the function f at $\mathbf{x}$ is nothing but the range in $\overline{\mathbb{R}}_P$ of the function f over the set $\{\mathbf{x}\}$, except when f is not defined at $\mathbf{x}$. In that case, the value set contains the single value NaN for Not-a-Number while the range is the empty set. It turns out that this distinction is crucial for our purposes.

But first, to recover the different definitions of interval division, we look at

$$\text{vset}(/, (X, Y)) \cap \mathbb{B} \qquad \mathbb{B} = \mathbb{R} \text{ or } \overline{\mathbb{R}}_A \qquad X, Y \in \mathbb{I}_r \tag{2.8}$$

where it is assumed that $\{\infty\} \cap \overline{\mathbb{R}}_A = \{\infty\} \cap \{-\infty, +\infty\} = \{-\infty, +\infty\}$. Let us start with a first simple example, the division $[1, 2] \div [0, 0]$. According to (2.8), we find for

- $\mathbb{B} = \mathbb{R}$: $\quad \text{vset}(/, ([1, 2], [0, 0])) \cap \mathbb{B} = \{\infty\} \cap \mathbb{R} = \emptyset = [1, 2] \div_F [0, 0]$
- $\mathbb{B} = \overline{\mathbb{R}}_A$:

$$\begin{aligned} \text{vset}(/, ([1, 2], [0, 0])) \cap \mathbb{B} &= \{\infty\} \cap \overline{\mathbb{R}}_A \\ &= \{-\infty, +\infty\} = \text{cset}(/, ([1, 2], [0, 0])) \end{aligned}$$

Before jumping to conclusions, we also consider the example $[0, 2] \div [0, 1]$. According to (2.8), we find for

- $\mathbb{B} = \mathbb{R}$

$$\begin{aligned} \text{vset}(/, ([0, 2], [0, 1])) \cap \mathbb{B} &= (\,[0, +\infty[\cup \{\infty\} \cup \{\tfrac{0}{0}\}) \cap \mathbb{R} \\ &= [0, +\infty[= [0, 2] \div_F [0, 1] \end{aligned}$$

- $\mathbb{B} = \overline{\mathbb{R}}_A$

$$\begin{aligned} \text{vset}(/, ([0, 2], [0, 1])) \cap \mathbb{B} &= (\,[0, +\infty[\cup \{\infty\} \cup \{\tfrac{0}{0}\}) \cap \overline{\mathbb{R}}_A \\ &= [0, +\infty[\cup \{-\infty, +\infty\} \subset \text{cset}(/, ([0, 2], [0, 1])) \end{aligned}$$

In general, functional and containment set interval division can be recovered from the value set for division in the following way. While this result is certainly not surprising, it will allow us to proceed with our own, vset-division of intervals in a natural way.

Theorem 2.1:

Let $X, Y \in \mathbb{I}_r$ be real, non-empty intervals, then

$$X \div_F Y = \text{vset}(/, (X, Y)) \cap \mathbb{R} \tag{2.9a}$$

$$X \div_C Y \equiv \text{cset}(/, (X, Y)) \supseteq \text{vset}(/, (X, Y)) \cap \overline{\mathbb{R}}_A \tag{2.9b}$$

The proof is given in Appendix A.

From *Theorem* 2.1 it is clear that, as expected, all real results of division are included in the result set, irrespective of the definition of interval division. When the mathematical result is not in $\mathbb{R}$ or is undefined, it is ignored in functional division. Another way to look at *Definition* 5, is to observe that when the mathematical result of division is projective infinity, this result is projected on the extended affine real line, while in cases where the result is undefined, the result is $\overline{\mathbb{R}}_A$. However, this interpretation of containment set is true only for division, not in general.

In fact, as long as intervals are defined as closed, connected subsets of $\mathbb{R}$ or $\overline{\mathbb{R}}_A$ and hence are uniquely characterized by exactly two (extended) reals, *Definition* 2 and *Definition* 5 are essentially the best we can do to deal with input arguments that contain points outside the domain of division. To introduce vset-interval arithmetic, we need an additional third parameter. This parameter comes from the unique partitioning of the value set in a subset of $\mathbb{R}$ and a subset containing "special" results. For rational functions with real coefficients, special means either infinity or undefined. We shall see in the next section that for arbitrary functions, the notion of special result is more general. Reconsidering the same examples as before, we find

$$\begin{aligned}
\mathrm{vset}(/, ([1,2],[0,0])) &= \{\infty\} = \emptyset \cup \{\infty\} \\
\mathrm{vset}(/, ([0,2],[0,1])) &= (\,[0,+\infty[\, \cup \{\infty\} \cup \{\frac{0}{0}\}) \\
&= [0,+\infty[\, \cup \{\infty, \mathrm{NaN}\} \\
\mathrm{vset}(/, ([1,2],[1,1])) &= [1,2] \cup \emptyset
\end{aligned}$$

In general, the value set of any rational function evaluated over $\mathbf{X} = (X_1, X_2) \subset \mathbb{R}^2$ consists on one hand of a subset of $\mathbb{R}$ and on the other hand of non-real results. For a rational function, the non-real results are a subset of the set

$$\mathbb{S} = \{\infty, \mathrm{NaN}\}$$

of special values. Since $\#2^{\mathbb{S}} = 4$, all we need to represent any non-real result of division is an additional two bits.

Definition 7

Let $\mathbb{I}_r$ be the set of real intervals and let $\mathbb{S}$ denote the set of special values. We define the set $\mathbb{I}_s$ of *vset-intervals* as

$$\mathbb{I}_s = \{X \cup s \mid X \in \mathbb{I}_r, s \in 2^{\mathbb{S}}\} \subset 2^{\mathbb{R} \cup \mathbb{S}} \tag{2.10}$$

For any $I = X \cup s \in \mathbb{I}_s$ we let

$$\begin{aligned}
\mathrm{Real}(I) &= X \in \mathbb{I}_r \\
\mathrm{Special}(I) &= s \in 2^{\mathbb{S}}
\end{aligned}$$

and refer to these as the real and special part of the vset-interval respectively.

Definition 8

Let $X, Y \in \mathbb{I}_r$ be real, non-empty intervals, then we define

$$\begin{aligned}
\div_V : \mathbb{I}_r^2 &\longrightarrow 2^{\mathbb{R} \cup \{\infty, \mathrm{NaN}\}} \\
(X, Y) &\longrightarrow X \div_V Y = \mathrm{vset}(/, (X, Y))
\end{aligned}$$

and the corresponding vset-interval division by

$$\begin{aligned}
/_V : \mathbb{I}_r^2 &\longrightarrow \mathbb{I}_s \\
(X, Y) &\longrightarrow X/_V Y = \mathrm{hull}_{\mathbb{I}_r}((X \div_V Y) \cap \mathbb{R}) \;\cup\; (X \div_V Y) \cap \mathbb{S}
\end{aligned}$$

We shall refer to $/_V$ as *value set* division or *vset*-division for short. It follows from *Definition* 8 and *Theorem* 2.1 that

$$X/_V Y = X/_F Y \;\cup\; \mathrm{Special}(X/_V Y)$$

In other words, $\text{Real}(X/_V Y)$ is precisely the result returned by functional division. Hence it follows from *Lemma* 2.1 and *Lemma* 2.2 that $\text{Real}(X/_V Y)$ is at least as sharp as the result returned by relational and containment set division.
A number of issues arise naturally with respect to the above definition.
First, what is the use of keeping track of non-real results? We shall answer thise question in a more general context in the next section and indicate that a generalization of vset-interval division to arbitrary functions f is important to obtain sharp, and especially correct, interval results (see *Theorem* 3.3 and *Theorem* 3.6 in *Section* 3).
Second, as we shall see in *Section* 4, keeping track of real and non-real results separately becomes very useful in a floating-point context because it makes it possible to distinguish between overflow and a true infinity result without raising exceptions. Let $\circledast$ denote the interval-rounded operation corresponding to $*$, then

$$2 \odot [1/2, \text{MaxFloat}] = [1, +\infty[$$

Here $+\infty$ is the floating-point representation of a value too large to be represented, while in

$$1 \oslash_V [0, 1] = [1, +\infty[\cup \{\infty\}$$

true mathematical infinity is always an element of the special part of the result and can hence not be confused with overflow.
Third, examples are given in [8, 12], to indicate that the sharpness of functional division is a problem for the application of the interval Newton method. That this is not a fundamental problem, is shown in [3] where the interval Newton method is reformulated to accomodate this remark.

Theorem 2.2: [3]

Let f be a continously differentiable function in $I \in \mathbb{I}_r$, let $x \in I$ and define

$$N_x := x - f(x)/f'(I) \ \in \mathbb{I} \tag{2.11}$$

If $0 \notin \{f(x)\} \cap f'(I)$, every zero z of f in I satisfies $z \in N_x \cap I$.

This formulation of Newton's method differs from the classical formulation in that the condition $0 \notin \{f(x)\} \cap f'(I)$ has been added. With this additional condition, the theorem holds for all definitions of interval division given here. Without it, the theorem only holds for relational and containment set division. Observe, however, that when $0 \in \{f(x)\} \cap f'(I)$, then $N_x =]-\infty, +\infty[$ according to relational division and $N_x = [-\infty, +\infty]$ according to containment set division. In that case the statement $z \in N_x \cap I = I$ is trivially satisfied and there is no contraction.
The additional condition in *Theorem* 2.2 doesn't make the theorem less general or less applicable. In an algorithmic implementation of Newton's method, whenever $N_x \cap I = I$, the interval I is bisected. In case the division operator / in (2.11) is relational or containment set interval division, the condition $0 \in \{f(x)\} \cap f'(I)$ automatically implies $N_x \cap I = I$. If we use either functional or value set division to implement Newton's method, this is not the case and the condition to start bisection should explicitly include $0 \in \{f(x)\} \cap f'(I)$.

3 Value sets and vset-interval arithmetic.

In the previous section we have seen that the concepts of containment set and value set extend the notion of real range of a function, in order to deal with, rather than ignore, points that are not in the domain of the function. Corresponding to the notion of range, containment and value set, there exist three interval division operations: functional division, based purely on the (real) range of the division operator, containment set division based on the notion of containment set and vset-division based on value sets.
While *Definition* 4 of containment set is given for arbitrary functions f, value sets are introduced in *Definition* 6 for rational functions only. In this section we extend the notion of value set for other

than rational functions. so that for any function f from $\overline{\mathbb{R}}_P^n$ to $\overline{\mathbb{C}}$, we can define its extension from $\mathbb{I}_s^n$ to $\mathbb{I}_s$.

Before giving the definition of value set for arbitrary functions f, consider the following example. The range of the function $\sqrt{\ }$ over the interval $[-1,1]$ is given by $R_{\sqrt{}} = [0,1] \cup \mathrm{i}[0,1]$. Now, remember that in a typed programming environment, there is no way to automatically switch from real to complex mode and hence we cannot enclose the result $R_{\sqrt{}}$. The functional and containment set approach ignore the complex part and return $R'_{\sqrt{}} = [0,1]$ as result, effectively loosing the essential property of containment. In our value set approach, we do not want to loose mathematical containment, and therefore return as result $R''_{\sqrt{}} = [0,1] \cup \{\mathrm{IaC}\}$. The abbreviation IaC stands for Is-a-Complex and is chosen in analogy with NaN for Not-a-Number. The semantics of $\{\mathrm{IaC}\}$ is that it represents a set of values V where $V \subseteq \mathbb{C}$. Here the subset $\{\mathrm{IaC}\}$ of the result $R''_{\sqrt{}}$ encloses all complex non-real results of $\sqrt{[-1,1]}$. While in this example it is also true that the set of values V represented by $\{\mathrm{IaC}\}$ satisfies $V \subseteq \mathbb{C} \setminus \mathbb{R}$, this is not the case in general, as will become clear from *Definition* 11. We formalize the semantics of the special value IaC in terms of sets by introducing a semantic function S.

Definition 9

Let $\mathbb{S} = \{\infty, \mathrm{IaC}, \mathrm{NaN}\}$ be the set of special values. For any $X \subseteq \mathbb{R} \cup \mathbb{S}$, we define the semantics $S(X)$ of X recursively as follows :

$$\begin{aligned} \forall x \in \mathbb{R} \cup \{\infty, \mathrm{NaN}\} : S(\{x\}) &= \{x\} \\ S(\{\mathrm{IaC}\}) &= \mathbb{C} \\ \forall X \subseteq \mathbb{R} \cup \mathbb{S} : S(X) &= \bigcup_{x \in X} S(\{x\}) \end{aligned}$$

Having introduced the special value IaC, we are now ready to define the value set of an arbitrary single-valued function.

Definition 10

Let f be a function of n variables and let $\mathbf{x} = (x_1, \ldots, x_n)$. The value set of the function $f : \overline{\mathbb{C}}^n \to \overline{\mathbb{C}}$ with domain D_f evaluated at the point $\mathbf{x}$ is denoted by $\mathrm{vset}(f, \{\mathbf{x}\})$ and is defined as follows:

$$\mathrm{vset}(f, \{\mathbf{x}\}) = \begin{cases} \{f(\mathbf{x})\} & f(\mathbf{x}) \in \overline{\mathbb{R}}_P \\ \{\mathrm{NaN}\} & \mathbf{x} \notin D_f \\ \{\mathrm{IaC}\} & f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

Since value sets (or, more precisely, their interval hull) propagate throughout the computation, we also need to define the value set of $\{\mathbf{x}\} = \{(x_1, \ldots, x_n)\}$ when at least one of the arguments x_i is NaN or IaC. While the definition of the former is obvious, this is not the case for the latter. The propagation of special values can be understood in terms of set arithmetic. Based on *Definition* 9, we could define

$$\mathrm{vset}(f, \{\mathrm{IaC}\}) = \bigcup_{x \in S(\{\mathrm{IaC}\})} \mathrm{vset}(f, \{x\}) = \bigcup_{x \in \mathbb{C}} \mathrm{vset}(f, \{x\})$$

There is a problem with this definition, however, in that it can blur explicit real-valued results, since in many cases $\mathbb{R} \subset \cup_{x \in \mathbb{C}} \mathrm{vset}(f, \{x\})$. Hence this approach needs to be refined. Intuitively, we return $\{\mathrm{IaC}\}$[2] whenever there exists at least one $\mathbf{z} \in \mathbb{C}^n$ such that $f(\mathbf{z}) \in \mathbb{C} \setminus \mathbb{R}$. On the other hand, if for all $\mathbf{z} \in \mathbb{C}^n$, $f(\mathbf{z}) \in \overline{\mathbb{R}}_P \cup \{\mathrm{NaN}\}$, it is possible to better quantify the (real-valued) results.

[2] As will become clear from *Definition* 11, we return the union of $\{\mathrm{IaC}\}$ and other special values in case there also exists at least one $\mathbf{z} \in \mathbb{C}^n$ such that $f(\mathbf{z})$ equals ∞ or such that $\mathbf{z} \notin D_f$.

Definition 11

Let D_f be the domain of a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Then we define

$$\text{vset}(f, \{\text{NaN}\}) = \{\text{NaN}\}$$

and

$$\text{vset}(f, \{\text{IaC}\}) = \begin{cases} \bigcup\limits_{z \in S(\{\text{IaC}\})} \text{vset}(f, \{z\}) & \forall z \in S(\{\text{IaC}\}) : f(z) \notin \mathbb{C} \setminus \mathbb{R} \\ \{\text{IaC}\} \cup \bigcup\limits_{\substack{z \in S(\{\text{IaC}\}) : z \notin D_f \\ z \in S(\{\text{IaC}\}) : f(z) = \infty}} \text{vset}(f, \{z\}) & \exists z \in S(\{\text{IaC}\}) : f(z) \in \mathbb{C} \setminus \mathbb{R} \end{cases}$$

This definition is generalized in a straightforward way to functions of n variables. Based on *Definition* 10 and *Definition* 11, we can now, as before, compute the value set of f for any subset $\mathbf{X} \subseteq (\mathbb{R} \cup \{\infty, \text{NaN}, \text{IaC}\})^n$ as follows:

$$\text{vset}(f, \mathbf{X}) = \bigcup_{\mathbf{x} \in \mathbf{X}} \text{vset}(f, \{\mathbf{x}\})$$

and hence are ready to introduce vset-interval arithmetic.
Before doing so, we illustrate the above definitions with some examples.
Let f be the modulus function. Then $\text{vset}(|\cdot|, \{\text{IaC}\}) = [0, +\infty[$. Similarly, if $f(x) = x \cdot 0$, then $\text{vset}(f, \{\text{IaC}\}) = 0$. On the other hand, if $f(x) = 1/x$ and $x = \text{IaC}$, there exists $z \in \mathbb{C}$ such that $1/z \in \mathbb{C} \setminus \mathbb{R}$ and therefore $\text{vset}(1/x, \{\text{IaC}\}) = \{\text{IaC}\} \cup \{\infty\}$[3]. In this case the set $\mathbb{R} \setminus \{0\} \subset \cup_{x \in \mathbb{C}} \text{vset}(f, \{x\})$ is not explicitly returned but contained in {IaC}. If this were not the case, all explicit real-valued results in expressions such as

$$\begin{aligned} \text{vset}(1/x, [1,2] \cup \{\text{IaC}\}) &= \text{vset}(1/x, [1,2]) \cup \text{vset}(1/x, \{\text{IaC}\}) \\ &= [1/2, 1] \cup \{\text{IaC}\} \cup \{\infty\} \end{aligned}$$

would be blurred. From these examples it should be clear that, unlike the value NaN, the value IaC, once generated, does not propagate infinitely through the computations but carries valuable information [4].
Based on the concept of value set, we are now ready to introduce vset-interval arithmetic, in the same way as interval arithmetic is based on the concept of range of the function, and the exception-free arithmetic in [12] is based on the concept of containment set. The advantage of basing interval arithmetic on the concept of value set rather than on the concept of range of a function, is that we are able to evaluate a function over an interval which contains points that are not in the domain of the real-valued function, and we do so without losing the containment property of interval arithmetic.

Definition 12

Let f be a function of n variables, let $\mathbb{S}$ be the set of special values

$$\mathbb{S} = \{\infty, \text{NaN}, \text{IaC}\} \qquad \#2^{\mathbb{S}} = 8$$

and let $\mathbb{I}_s$ be the set of vset-intervals defined by (2.10). Then the vset-extension of f, which we denote by f_V, is given by

$$\begin{aligned} f_V : \mathbb{I}_s^n &\longrightarrow \mathbb{I}_s \\ \mathbf{I} &\longrightarrow f_V(\mathbf{I}) = \text{Real}(f_V(\mathbf{I})) \cup \text{Special}(f_V(\mathbf{I})) \end{aligned}$$

[3] In the actual implementation, we support the special value {IaC} as well as the special value $\{\text{IaC}_0\} \equiv \{\text{IaC}\} \setminus \{0\}$. The distinction between {IaC} and $\{\text{IaC}_0\}$ is especially relevant for division since $\text{vset}(1/x, \{\text{IaC}_0\}) = \{\text{IaC}_0\}$.

where

$$\text{Real}(f_V(\mathbf{I})) = \text{hull}_{\mathbb{I}_r}(\text{vset}(f,\mathbf{I}) \setminus \mathbb{S})$$
$$\text{Special}(f_V(\mathbf{I})) = \text{vset}(f,\mathbf{I}) \cap \mathbb{S}$$

Note that to determine the real part of the result $f_V(\mathbf{I})$, we now compute $\text{vset}(f,\mathbf{I}) \setminus \mathbb{S}$ rather than $\text{vset}(f,\mathbf{I}) \cap \mathbb{R}$ in *Definition* 8. Whenever $\text{IaC} \notin \text{vset}(f,\mathbf{I})$ both these expressions are equivalent. When $\text{IaC} \in \text{vset}(f,\mathbf{I})$, it is not possible to quantify $\{\text{IaC}\} \cap \mathbb{R}$ without overestimation, since all we know is that $\{\text{IaC}\}$ represents a set of values V where $V \subseteq \mathbb{C}$ and $V \cap (\mathbb{C} \setminus \mathbb{R}) \neq \emptyset$.
We further remark that in general $f_V(\mathbf{I}) \supseteq \text{vset}(f,\mathbf{I})$, where the latter is a subset of $\mathbb{R} \cup \mathbb{S}$, while $f_V(\mathbf{I})$ is a vset-interval. To represent the special part of $f_V(\mathbf{I})$, three rather than two bits are now needed, in addition to the two real numbers to represent the real part of the vset-interval. This is a small overhead in representation and in computation, compared to traditional and containment set interval arithmetic, but it guarantees containment and sharpness in all situations, as the next theorem indicates. This theorem also generalizes the result for division, given in *Theorem* 2.1, to arbitrary functions f.

Theorem 3.3:

Let $f : \mathbb{R}^n \to \overline{\mathbb{C}}$ be a function of n variables with domain D_f, let $\overline{D}_f$ denote the closure of D_f and let S be the semantic function introduced in *Definition* 9. For $\mathbf{x} \notin D_f$ we set $f(\mathbf{x}) = \text{NaN}$. The vset-extension f_V of f is such that for any $\mathbf{I} = (I_1, \ldots, I_n) \in \mathbb{I}_r^n$

$$\mathbf{x} \in \mathbf{I} \Rightarrow f(\mathbf{x}) \in S(f_V(\mathbf{I})) \tag{3.1}$$

Furthermore, the relationship given in *Table* 1 between containment set and value set holds. With the exception of (c) and (f):

$$\text{cset}(f,\mathbf{I}) \supseteq \text{vset}(f,\mathbf{I}) \cap \overline{\mathbb{R}}_A \tag{3.2}$$

(a)	$\mathbf{x} \in D_f$	$f(\mathbf{x}) \in \mathbb{R}$	$\text{cset}(f,\{\mathbf{x}\}) \supseteq \text{vset}(f,\{\mathbf{x}\}) = \{f(\mathbf{x})\}$
(b)		$f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R}$	$\text{cset}(f,\{\mathbf{x}\}) \subseteq \overline{\mathbb{R}}_A, \text{vset}(f,\{\mathbf{x}\}) = \{\text{IaC}\}$
(c)		$f(\mathbf{x}) = \infty$	$\text{cset}(f,\{\mathbf{x}\}) \subseteq \overline{\mathbb{R}}_A, \text{vset}(f,\{\mathbf{x}\}) = \{f(\mathbf{x})\}$
(d)	f continuous in $\mathbf{x}$	$f(\mathbf{x}) \in \mathbb{R}$	$\text{cset}(f,\{\mathbf{x}\}) = \text{vset}(f,\{\mathbf{x}\}) = \{f(\mathbf{x})\}$
(e)	f continuous in $\mathbf{x}$	$f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R}$	$\text{cset}(f,\{\mathbf{x}\}) = \emptyset, \text{vset}(f,\{\mathbf{x}\}) = \{\text{IaC}\}$
(f)	f continuous in $\mathbf{x}$	$f(\mathbf{x}) = \infty$	$\text{cset}(f,\{\mathbf{x}\}) \subseteq \{-\infty,+\infty\}, \text{vset}(f,\{\mathbf{x}\}) = \{f(\mathbf{x})\}$
(g)	$\mathbf{x} \in \overline{D}_f \setminus D_f$		$\text{cset}(f,\{\mathbf{x}\}) \subseteq \overline{\mathbb{R}}_A, \text{vset}(f,\{\mathbf{x}\}) = \{\text{NaN}\}$
(h)	$\mathbf{x} \notin \overline{D}_f$		$\text{cset}(f,\{\mathbf{x}\}) = \emptyset, \text{vset}(f,\{\mathbf{x}\}) = \{\text{NaN}\}$

Table 1

The proof is given in Appendix A. The fact that (3.2) doesn't hold in the cases (c) and (f) is illustrated in the following examples:
(c) For $f(x) = 1/\text{ceil}(|x|)$, we find $\text{cset}(f,\{0\}) = \{1\}$ while $\text{vset}(f,\{0\}) = \{\infty\} = \{f(0)\}$. Note that in this case the essential property of containment is violated by the cset approach, since $\text{cset}(f,\{0\}) \cap (\{f(0)\} \cap \overline{\mathbb{R}}_A) = \emptyset$!
(f) Let $f(x) = 1/x^2$. Then $\text{cset}(f,\{0\}) = \{+\infty\}$, while $\text{vset}(1/x^2,\{0\}) = \{\infty\}$. Here $\text{cset}(f,\{0\}) \subset \text{vset}(f,\{0\}) \cap \overline{\mathbb{R}}_A$.
The next examples illustrate the cases (a) and (g) in *Table* 1:
(a) Consider the function $f(x) = \text{floor}(x)$. Then $\text{cset}(f,\{1\}) = \{0,1\} \supset \text{vset}(f,\{1\}) = \{1\}$.

(g) For $f(x) = (x^2 - 4)/(x + 2)$ we find $\text{cset}(f, \{-2\}) = \{-4\}$ and $\text{vset}(f, \{-2\}) = \{\text{NaN}\}$.

It follows from *Table* 1 that, whenever f is evaluated over a subset $\mathbf{X} \subset \mathbb{R}^n$ such that $f(\mathbf{X}) \cap \mathbb{C} \setminus \mathbb{R} \neq \emptyset$, containment is only achieved by the value set approach. An example of this situation was given in *Section* 1. Another example where lack of containment leads to erroneous conclusions, occurs in the application of Brouwer's fixpoint theorem [6]. We first recall this important theorem.

Theorem 3.4: [11, P. 4]

Let $\mathbf{X}$ be a nonempty, convex and compact subset of $\mathbb{C}^n$. Let $f : \mathbf{X} \to \mathbb{C}^n$ be a continuous function and $\widetilde{f} : 2^{\mathbf{X}} \to 2^{\mathbb{C}^n}$ be its set extension with $f(x) \in \widetilde{f}(A)$ for all $x \in A$ and $A \subseteq \mathbf{X}$. If

$$\widetilde{f}(\mathbf{X}) \subseteq \mathbf{X}$$

then the equation $f(\mathbf{x}) = \mathbf{x}$ has at least one solution in $\mathbf{X}$.

Now consider the function

$$f(x) = \sqrt{x} - 1$$

which has no real fixpoint and the iteration

$$X_{k+1} = \widetilde{f}(X_k) = \sqrt{X_k} - 1 \qquad k = 0, 1, \ldots$$

For $X_0 = [-4, 1]$ we find, when we compute $\widetilde{f}$ according to the concepts of range and of containment set, that

$$X_1 = \sqrt{[-4, 1]} - 1 = [0, 1] - 1 = [-1, 0] \subset X_0$$

and hence $\widetilde{f}(X_0) \subset X_0$ while f has no fixpoint in X_0! Computing $\widetilde{f}$ according to vset-interval arithmetic, we find

$$X_1 = \sqrt{[-4, 1]} - 1 = [0, 1] \cup \{\text{IaC}\} - 1 = [-1, 0] \cup \{\text{IaC}\} \not\subset X_0$$

and no incorrect conclusions are possible. It is true in general that, unlike the other approaches, the vset-interval approach can not lead to erroneous conclusions in the application of Brouwer's fixpoint theorem. Like the other approaches, however, it can suffer from overestimation of the set extension $\widetilde{f}$ of f. In such cases, no (and hence no erroneous) conclusions can be drawn about the existence of a fixpoint, as the following example illustrates. Consider the function

$$f(x) = \frac{\text{Log}^2(x)}{\pi^2}$$

which has the fixpoint $x = -1$ and the iteration

$$X_{k+1} = \frac{\text{Log}^2(X_k)}{\pi^2} \qquad k = 0, 1, \ldots$$

For $X_0 = [-1, -1]$, we find when we compute an enclosure for $\widetilde{f}$ in vset-interval arithmetic, that

$$X_1 = \emptyset \cup \text{IaC} \not\subseteq X_0$$

Due to overestimation of $\widetilde{f}(X_0)$, we cannot infer that the equation $f(x) = x$ has a fixpoint in X_0.

We conclude this section with two important results. The first is the inclusion isotonicity of vset-interval arithmetic.

Theorem 3.5: (INCLUSION ISOTONICITY)

Let $\mathbf{X} = X_1 \times \ldots \times X_n$ and $\mathbf{Y} = Y_1 \times \ldots \times Y_n$ both be subsets of $(\mathbb{R} \cup \mathbb{S})^n$ and let $f : \overline{\mathbb{C}}^n \to \overline{\mathbb{C}}$. The following holds :

$$\mathbf{X} \subseteq \mathbf{Y} \Rightarrow f_V(\mathbf{X}) \subseteq f_V(\mathbf{Y})$$

This result follows from *Definition* 12 of vset-extension.
Second, we state the inclusion property of vset-interval arithmetic. This is related to the effect of breaking up an interval expression into component subexpressions, and evaluating the vset-extension of each subexpression.

Theorem 3.6: (INCLUSION PROPERTY)

Let g be a function from $\overline{\mathbb{C}}^n$ to $\overline{\mathbb{C}}^m$ and let f be a function from $\overline{\mathbb{C}}^m$ to $\overline{\mathbb{C}}$. For any $\mathbf{X} \subseteq \overline{\mathbb{C}}^m$, we set $\widetilde{f}(\mathbf{X}) = \{f(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}\}$. Then, for any $\mathbf{I} \in \mathbb{I}_s^n$, the following holds :

$$\widetilde{f \circ g}(S(\mathbf{I})) \subseteq S(f_V(g_V(\mathbf{I}))) \tag{3.3}$$

where S is the semantic function given in *Definition* 9.

This result, which is proved in Appendix A, is noteworthy in the following sense. When the functions f and g are real-valued, (3.3) trivially holds if $\mathbf{I} \subseteq D_g$ and $g(\mathbf{I}) \subseteq D_f$. In that case, the vset-extension is nothing but the interval hull of the range of the function. However, the importance of this theorem is that the result continues to hold, also when the input arguments contain points outside the domain of the underlying point function, considered as a function from $\overline{\mathbb{R}}_P$ to $\overline{\mathbb{R}}_P$. Consider $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = x$. For $I = [-1, -1]$ we find

$$\begin{aligned} \widetilde{f \circ g}([-1,-1]) = [-1,-1] &\subseteq S(f_V(g_V([-1,-1]))) \\ &= S(f_V(\emptyset \cup \{\mathrm{IaC}\})) = S(\emptyset \cup \{\mathrm{IaC}\}) = \mathbb{C} \end{aligned}$$

In this example, the inclusion property doesn't hold anymore if in (3.3) we replace the vset-extension of f and g by the containment set of f and g. Indeed,

$$\widetilde{f \circ g}([-1,-1]) = [-1,-1] \not\subseteq \mathrm{cset}(f, \mathrm{cset}(g, [-1,-1])) = \mathrm{cset}(f, \emptyset) = \emptyset$$

since $[-1, -1]$ is not in the closure of the domain of g, considered as a function from $\overline{\mathbb{R}}_P$ to $\overline{\mathbb{R}}_P$.
The above theorem testifies to the purpose and achievement of the value set approach: it carries sufficient information to be mathematically reliable in all situations where the input arguments contain points outside the domain of the underlying function.

4 Floating-point implementation of vset-interval arithmetic.

The previous section deals with real vset-interval arithmetic. That is, for each $I \in \mathbb{I}_s$, the endpoints of $\mathrm{Real}(I)$ are real. Let $\mathbb{F}$ denote the set of floating-point numbers. We obtain floating-point vset-intervals if in

$$\{X \cup s \mid X \in \mathbb{I}_r, s \in 2^{\mathbb{S}}\} \tag{4.1}$$

we replace $\mathbb{I}_r$ by

$$\{[a,b] \mid a,b \in \mathbb{F}\} \cup \{]-\infty, b] \mid b \in \mathbb{F}\} \cup \{[a, +\infty[\mid a \in \mathbb{F}\} \cup \{]-\infty, +\infty[\} \tag{4.2}$$

In other words, a floating-point vset-interval I is represented by two floating-point numbers $\inf(I)$ and $\sup(I)$ which determine $\mathrm{Real}(I)$ and a three-bit structure which represents $\mathrm{Special}(I)$. The results in the previous sections continue to hold for floating-point vset-intervals if we add outward interval rounding to each operation. However, if in (4.1) we replace $\mathbb{I}_r$ by a set $\mathbb{I}_{\mathbb{F}}$ which is even more expressive than (4.2), we can obtain sharper floating-point vset-interval results. In the definition of the set $\mathbb{I}_{\mathbb{F}}$ we shall take advantage of the fact that any floating-point implementation which complies with the IEEE 754 standard [1] has two representations for zero, in casu $+0$ and -0.

Definition 13

Let $\mathbb{F}$ denote the set of IEEE floating-point representations. Let f_1, f_2 denote two finite, non-zero floating-point numbers and let f denote any floating-point representation different from NaN. We define the set $\mathbb{I}_{\mathbb{F}}$ of floating-point intervals by

$$\mathbb{I}_{\mathbb{F}} = \{\langle x_1, x_2\rangle \mid x_1, x_2 \in \mathbb{F}\}$$

where the semantics of $\langle x_1, x_2\rangle$ is given by

Syntax		Semantics
$\langle f_1, f_2\rangle$	$\equiv$	$[f_1, f_2]$
$\langle f_1, +\infty\rangle$	$\equiv$	$[f_1, +\infty[$
$\langle -\infty, f_2\rangle$	$\equiv$	$]-\infty, f_2]$
$\langle -\infty, +\infty\rangle$	$\equiv$	$]-\infty, +\infty[$
$\langle f_1, +0\rangle$	$\equiv$	$[f_1, 0]$
$\langle -0, f_2\rangle$	$\equiv$	$[0, f_2]$
$\langle -0, +0\rangle$	$\equiv$	$[0, 0]$
$\langle f_1, -0\rangle$	$\equiv$	$[f_1, 0[$
$\langle +0, f_2\rangle$	$\equiv$	$]0, f_2]$
$\langle -0, +\infty\rangle$	$\equiv$	$[0, +\infty[$
$\langle +0, +\infty\rangle$	$\equiv$	$]0, +\infty[$
$\langle -\infty, +0\rangle$	$\equiv$	$]-\infty, 0]$
$\langle -\infty, -0\rangle$	$\equiv$	$]-\infty, 0[$
$\langle \mathrm{NaN}, \mathrm{NaN}\rangle$	$\equiv$	$\emptyset$
$\langle \mathrm{NaN}, f\rangle$		undefined
$\langle f, \mathrm{NaN}\rangle$		undefined

The interpretation of the angle brackets in *Definition* 13 as [or] depends on the value of the left/right endpoint and can be understood intuitively. Indeed, in a floating-point context, the left endpoint of Real(I) is always the result of a round down operation, and the right endpoint always the result of a round up operation. Since $+0$ as left endpoint can only occur as the result of rounding down a non-zero positive number smaller than the smallest unnormalized number, it is natural to interpret $\langle +0, f\rangle$ as $]0, f]$. If the left endpoint is -0, the interval result contains the mathematical zero and $\langle -0, f\rangle$ is to be interpreted as $[0, f]$.
Based on the sets $\mathbb{I}_{\mathbb{F}}$ and $\mathbb{S} = \{\infty, \mathrm{IaC}_0, \mathrm{IaC}, \mathrm{NaN}\}$ for the real and special part respectively, the set $\mathbb{I}_s(\mathbb{F})$ of floating-point vset-intervals is now given by

$$\mathbb{I}_s(\mathbb{F}) = \{X \cup s \mid X \in \mathbb{I}_{\mathbb{F}}, s \in 2^{\mathbb{S}}\} \qquad \mathbb{S} = \{\infty, \mathrm{IaC}_0, \mathrm{IaC}, \mathrm{NaN}\} \tag{4.3}$$

Note that, as indicated in the previous section, our implementation of vset-interval arithmetic supports, besides the special value IaC, also the special value IaC_0 which can be interpreted as 'a set of nonzero complex numbers'. The addition of the special value IaC_0 implies that we need to refine the definition of value set and the propagation of special values. This can be done in a straightforward way and is detailed in Appendix B.
The advantages of using $\mathbb{I}_{\mathbb{F}}$ to define floating-point vset-intervals rahter than (4.2), are plentiful. First, we observe that in any floating-point set $\mathbb{F}$, the representations $\pm\infty$ have two interpretations: $\pm\infty$ can represent both overflow and an infinite result. We unravel this two-fold interpretation in floating-point vset-intervals: true infinite results are represented by the special value ∞, while overflow is represented by an infinite endpoint in the real part of the vset-interval. Such a distinction has advantages. We find for example that $\sin([0, +\infty[) = [-1, +1]$, while $\sin([-\pi, \pi] \cup \{\infty\}) = [-1, +1] \cup \{\mathrm{NaN}\}$ since sin of any finite number lies in the interval $[-1, +1]$ while $\sin(\infty)$ is mathematically undefined. Moreover, it guarantees exception-free interval arithmetic, since there is no need for overflow, underflow, invalid and zero-divide flags.
Similarly, in floating-point arithmetic, the representations ± 0 have two interpretations: they can represent both zero as well as underflow (a very small, non-zero result). It is clear that the situation

at 0 and the situation at ∞ are dual and therefore we need to be able to represent intervals which include/exclude 0 as endpoint. That is precisely what we achieve with *Definition* 13. As an example, we find $1/[1,+\infty[=]0,1] \equiv \langle +0, 1\rangle$, while $1/([1,+\infty[\cup\{\infty\}) = [0,1] \equiv \langle -0, 1\rangle$.
This example shows the benefits of making a distinction between very small nonzero numbers, and zero itself. If the result of an operation is different from zero, we can represent this information to get a sharp result. If the result contains zero, we don't ignore possibly invalid or infinite results later on in the computation. Interval systems which don't provide a representation for half open intervals at 0, cannot make this distinction.
Finally, some excerpts of the tables, detailing the basic arithmetic operations on vset-intervals, are given. We assume that $x_1, x_2 \in \mathbb{R}^+$ and $x_2 \neq 0$.

$+$	$]0, x_2]$	$[0, x_2]$	$[x_1, +\infty[$	$[-x_1, +\infty[\cup \{\infty\}$
$]0, x_2]$	$]0, 2x_2]$	$]0, 2x_2]$	$[x_1, +\infty[$	$[-x_1, +\infty[\cup \{\infty\}$
$[0, x_2]$	$]0, 2x_2]$	$[0, 2x_2]$	$[x_1, +\infty[$	$[-x_1, +\infty[\cup \{\infty\}$
$[x_1, +\infty[$	$[x_1, +\infty[$	$[x_1, +\infty[$	$[2x_1, +\infty[$	$[2x_1, +\infty[\cup \{\infty\}$
$[-x_1, +\infty[\cup \{\infty\}$	$[-x_1, +\infty[\cup \{\infty\}$	$[-x_1, +\infty[\cup \{\infty\}$	$[2x_1, +\infty[\cup \{\infty\}$	$[2x_1, +\infty[\cup \{\infty, \mathrm{NaN}\}$

$\times$	$]0, x_2]$	$[0, x_2]$	$[x_1, +\infty[$	$[-x_1, +\infty[\cup \{\infty\}$
$]0, x_2]$	$\left]0, x_2^2\right]$	$\left[0, x_2^2\right]$	$]0, +\infty[$	$[-x_1x_2, +\infty[\cup \{\infty\}$
$[0, x_2]$	$\left[0, x_2^2\right]$	$\left[0, x_2^2\right]$	$[0, +\infty[$	$[-x_1x_2, +\infty[\cup \{\infty, \mathrm{NaN}\}$
$[x_1, +\infty[$	$]0, +\infty[$	$[0, +\infty[$	$\left[x_1^2, +\infty\right[$	$]-\infty, +\infty] \cup \{\infty\}$
$[-x_1, +\infty[\cup \{\infty\}$	$[-x_1x_2, +\infty[\cup \{\infty\}$	$[-x_1x_2, +\infty[\cup \{\infty, \mathrm{NaN}\}$	$]-\infty, +\infty] \cup \{\infty\}$	$]-\infty, +\infty] \cup \{\infty, \mathrm{NaN}\}$

$/$	$]0, x_2]$	$[0, x_2]$	$[x_1, +\infty[$	$[-x_1, +\infty[\cup \{\infty\}$
$]0, x_2]$	$]0, +\infty[$	$]0, +\infty[\cup \{\infty\}$	$]0, x_1/x_2]$	$]-\infty, +\infty] \cup \{\infty\}$
$[0, x_2]$	$[0, +\infty[$	$[0, +\infty[\cup \{\infty, \mathrm{NaN}\}$	$[0, x_1/x_2[$	$]-\infty, +\infty] \cup \{\infty, \mathrm{NaN}\}$
$[x_1, +\infty[$	$[x_1/x_2, +\infty[$	$[x_1/x_2, +\infty[\cup \{\infty\}$	$]0, +\infty[$	$]-\infty, +\infty] \cup \{\infty\}$
$[-x_1, +\infty[\cup \{\infty\}$	$]-\infty, +\infty] \cup \{\infty\}$	$]-\infty, +\infty] \cup \{\infty, \mathrm{NaN}\}$	$[-1, +\infty[\cup \{\infty\}$	$]-\infty, +\infty] \cup \{\infty, \mathrm{NaN}\}$

Floating-point vset-interval arithmetic as described here is implemented as part of the larger Arithmos environment [2]. The interested reader can find reports on the project's progress and download software demonstrations at [2].

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Appendix A. Proofs of the lemmas and theorems

A.1. **Notation.** In order to prove the theorems in the paper, we introduce some additional notation listed in Table A.1.

X_0	$X \setminus \{0\}$
$(a_n)_n$	the sequence $(a_1, a_2, \ldots)$
$(a_{k_n})_n$	the subsequence $(a_{k_1}, a_{k_2}, \ldots)$ of $(a_n)_n$
$(a_n)_n \to a$	$(a_n)_n$ converges to a
$(a_n)_n \dashv a$	a is an accumulation point of $(a_n)_n$

Table A.1 Notation

A.2. **Some additional lemmas.** We need several additional lemmas to prove the theorems in the paper.

Lemma A.1. *If* $X \subseteq \mathbb{R}$, *then* $\mathrm{hull}_{\mathbb{I}_r}(X) = \mathrm{hull}_{\mathbb{I}_e}(X) \cap \mathbb{R}$

Proof. By definition. □

Lemma A.2. *If* $X, Y_1, Y_2, Y \subseteq \mathbb{R}$ *and* $Y_1 \cup Y_2 = Y$, *then*

$$(X \div_R Y_1) \cup (X \div_R Y_2) = X \div_R Y$$
$$(X \div_C Y_1) \cup (X \div_C Y_2) = X \div_C Y$$
$$(X \div_V Y_1) \cup (X \div_V Y_2) = X \div_V Y$$

Proof. Trivial. □

Corollary A.3. *If* $X, Y_1, Y_2, Y \in \mathbb{I}_r$ *and* $Y_1 \cup Y_2 = Y$, *then*

$$(X/_R Y_1) \cup (X/_R Y_2) \subseteq X/_R Y$$
$$(X/_C Y_1) \cup (X/_C Y_2) \subseteq X/_C Y$$

Lemma A.4. *If* $X \subseteq \mathbb{R}$ *and* $Y \subseteq \mathbb{R}_0$, *then*

$$(X \div_F Y) = (X \div_R Y) = (X \div_C Y) \subseteq \mathbb{R}$$

Proof. It follows easily from

$$\forall x, y, z \in \mathbb{R}, y \neq 0 : z = \frac{x}{y} \iff x = yz$$

that $(X \div_F Y) = (X \div_R Y) \subseteq \mathbb{R}$ if $0 \notin Y$. We next prove that $X \div_C Y = X \div_R Y$. Suppose that $z \in X \div_R Y$. Then there exist $x \in X$ and $y \in Y$ such that $x = yz$. Because $y \neq 0$, it follows that $z = \frac{x}{y}$. Now the constant sequence $((x, y))_n$ converges to (x, y), and the sequence $\left(\frac{x}{y}\right)_n$ converges to $\frac{x}{y} = z$. So $z \in \mathrm{cset}(/, (X, Y)) \equiv X \div_C Y$.
If on the other hand $z \in X \div_C Y$, then there exists a real sequence $((x_n, y_n))_n$ such that $((x_n, y_n))_n \to (x, y) \in X \times Y$ and $\left(\frac{x_n}{y_n}\right)_n \dashv z$. We can find a subsequence $\left(\frac{x_{k_n}}{y_{k_n}}\right)_n$ such that $\left(\frac{x_{k_n}}{y_{k_n}}\right)_n \to z$ and then we still have that $((x_{k_n}, y_{k_n}))_n \to (x, y)$. Since $y \neq 0$, the division is continuous in (x, y) and thus $\frac{x}{y} = z$, or equivalent $x = yz$. We know that $x \in X$ and $y \in Y$, so $z \in X \div_R Y$. □

Lemma A.5. *If $\emptyset \neq X \subseteq \mathbb{R}$, the following holds:*

$$X \div_R \{0\} = \begin{cases} \emptyset & \text{if } 0 \notin X \\ \mathbb{R} & \text{if } 0 \in X \end{cases}$$

$$X \div_C \{0\} = \begin{cases} \{-\infty, +\infty\} & \text{if } 0 \notin X \\ \overline{\mathbb{R}}_A & \text{if } 0 \in X \end{cases}$$

$$X \div_V \{0\} \subseteq \{\infty, \mathrm{NaN}\}$$

Proof. By definition of relational, containment and value set division. □

A.3. **Proof of lemma 2.2.**

Lemma. *Let $X, Y \in \mathbb{I}_r$, then*

$$X \div_R Y = (X \div_C Y) \cap \mathbb{R} \tag{1}$$

and

$$X /_R Y \subseteq (X /_C Y) \cap \mathbb{R} \tag{2}$$

Proof. If $0 \notin Y$, (1) immediately follows from lemma A.4 while if $0 \in Y$, it is implied by lemmas A.5 and A.2. We furhter have that

$$\begin{aligned} X /_R Y &= \mathrm{hull}_{\mathbb{I}_r}(X \div_R Y) \\ &= \mathrm{hull}_{\mathbb{I}_e}(X \div_R Y) \cap \mathbb{R} && \text{by lemma A.1} \\ &\subseteq \mathrm{hull}_{\mathbb{I}_e}(X \div_C Y) \cap \mathbb{R} && \text{by (1)} \\ &= (X /_C Y) \cap \mathbb{R} \end{aligned}$$

which completes the proof. □

A.4. **Proof of theorem 2.1.**

Theorem. *Let $X, Y \in \mathbb{I}_r$ be real, non-empty intervals, then*

$$X \div_F Y = (X \div_V Y) \cap \mathbb{R} \equiv \mathrm{vset}(/, (X, Y)) \cap \mathbb{R} \tag{3}$$

$$X \div_C Y \supseteq (X \div_V Y) \cap \overline{\mathbb{R}}_A \equiv \mathrm{vset}(/, (X, Y)) \cap \overline{\mathbb{R}}_A \tag{4}$$

Proof. The first statement of the theorem follows directly from the definition, since

$$X \div_F Y = \left\{ \frac{x}{y} \in \mathbb{R} \mid x \in X, y \in Y \right\}$$

$$X \div_V Y = \left\{ \frac{x}{y} \in \overline{\mathbb{R}}_P \mid x \in X, y \in Y \right\} \cup \{\mathrm{NaN} \mid (0,0) \in X \times Y\}$$

We know from lemma A.4 that $X \div_C Y_0 = X \div_V Y_0 \subseteq \mathbb{R}$ which proves (4) if $0 \notin Y$. If $0 \in Y$, then according to lemma A.5, we have that

$$(X \div_V \{0\}) \cap \overline{\mathbb{R}}_A \subseteq \{-\infty, +\infty\} \subseteq X \div_C \{0\}$$

which, taking into account lemma A.2, completes the proof of the theorem. □

A.5. **Proof of theorem 3.3.**

Theorem. *Let $f : \mathbb{R}^n \to \overline{\mathbb{C}}$ be an n-ary function with domain D_f and let $\overline{D}_f$ denote the closure of D_f. The following holds :*

(a) $\forall \mathbf{x} \in D_f : f(\mathbf{x}) \in \mathbb{R} \Rightarrow \mathrm{cset}(f, \{\mathbf{x}\}) \supseteq \mathrm{vset}(f, \{\mathbf{x}\})$

Proof. We construct the constant sequence $(\mathbf{x})_n$. Because $(\mathbf{x})_n \to \mathbf{x} \in \mathbf{I}$ and $(f(\mathbf{x}))_n \to f(\mathbf{x})$, we see that $f(\mathbf{x}) \in \mathrm{cset}(f, \{\mathbf{x}\})$. □

(b) $\forall \mathbf{x} \in D_f : f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R} \Rightarrow \mathrm{cset}(f, \{\mathbf{x}\}) \subseteq \overline{\mathbb{R}}_A, \mathrm{vset}(f, \{\mathbf{x}\}) = \{\mathrm{IaC}\}$

Proof. By definition. □

(c) $\forall \mathbf{x} \in D_f : f(\mathbf{x}) = \infty \Rightarrow \mathrm{cset}(f, \{\mathbf{x}\}) \subseteq \overline{\mathbb{R}}_A, \mathrm{vset}(f, \{\mathbf{x}\}) = \{\infty\}$

Proof. By definition. □

(d) If f is continuous in $\mathbf{x} \in D_f$ and $f(\mathbf{x}) \in \mathbb{R}$, then $\mathrm{cset}(f, \{\mathbf{x}\}) = \mathrm{vset}(f, \{\mathbf{x}\})$

Proof. From the definition of value set, it is clear that $\mathrm{vset}(f, \{\mathbf{x}\}) = \{f(\mathbf{x})\}$.

Suppose now that $y \in \mathrm{cset}(f, \{\mathbf{x}\})$, then there exists a real sequence $(\mathbf{x}_n)_n$, such that $(\mathbf{x}_n)_n \to \mathbf{x}$ and $(f(\mathbf{x}_n))_n \dashv y$. We can find a subsequence $(\mathbf{x}_{k_n})_n$ such that $(f(\mathbf{x}_{k_n}))_n \to y$, and then we still have that $(\mathbf{x}_{k_n})_n \to \mathbf{x}$. Since f is continuous in $\mathbf{x}$, it follows that $f(\mathbf{x}) = y$. □

(e) If f is continuous in $\mathbf{x} \in D_f$ and $f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R}$, then

$$\mathrm{cset}(f, \{\mathbf{x}\}) = \emptyset, \mathrm{vset}(f, \{\mathbf{x}\}) = \{\mathrm{IaC}\}$$

Proof. The fact that $\mathrm{vset}(f, \{\mathbf{x}\}) = \{\mathrm{IaC}\}$ follows directly from the definition. Now assume that $y \in \mathrm{cset}(f, \{\mathbf{x}\})$. Since $f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R}$, we know that $\Im(f(\mathbf{x})) \neq 0$. Now choose $0 < \varepsilon < \frac{\Im(f(\mathbf{x}))}{2}$.

Since f is continuous in $\mathbf{x}$, we can find δ such that

$$\forall \mathbf{z} \in D_f : |\mathbf{z} - \mathbf{x}| \leq \delta \Rightarrow |f(\mathbf{z}) - f(\mathbf{x})| \leq \varepsilon \tag{5}$$

Because $y \in \mathrm{cset}(f, \{\mathbf{x}\})$, there exists a sequence $(\mathbf{x}_n)_n$ which converges to $\mathbf{x}$ such that $(f(\mathbf{x}_n))_n \dashv y$. Since $(\mathbf{x}_n)_n \to \mathbf{x}$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n > n_0 : |\mathbf{x} - \mathbf{x}_n| \leq \delta \tag{6}$$

The fact that $(f(\mathbf{x}_n))_n \dashv y$ allows us to find some $n_1 \in \mathbb{N}$ such that $n_1 > n_0$ and

$$|y - f(\mathbf{x}_{n_1})| \leq \varepsilon \tag{7}$$

Putting $\mathbf{z} = \mathbf{x}_{n_1}$ in (5), we find

$$|f(\mathbf{x}_{n_1}) - f(\mathbf{x})| \leq \varepsilon \tag{8}$$

and hence $|y - f(\mathbf{x})| \leq 2\varepsilon$. This is impossible, since $y \in \mathbb{R}$, $f(\mathbf{x}) \in \mathbb{C} \setminus \mathbb{R}$, and $\Im(f(\mathbf{x})) > 2\varepsilon$. So our assumption that $y \in \mathrm{cset}(f, \{\mathbf{x}\})$ must be incorrect. □

(f) If f is continuous in $\mathbf{x} \in D_f$ and $f(\mathbf{x}) = \infty$, then $\mathrm{cset}(f, \{\mathbf{x}\}) \subseteq \{-\infty, +\infty\}$ and $\mathrm{vset}(f, x) = \{\infty\}$

Proof. By definition $\mathrm{vset}(f, x) = \{\infty\}$. Now suppose there is an element $y \in \mathrm{cset}(f, \{\mathbf{x}\})$. Again we create a sequence $(\mathbf{x}_n)_n$ such that $(\mathbf{x}_n)_n \to \mathbf{x}$ and $(f(\mathbf{x}_n))_n \dashv y$. And again we take a subsequence $(f(\mathbf{x}_{k_n}))_n \to y$. Since f is continuous in $\mathbf{x}$, and because $(\mathbf{x}_{k_n})_n \to \mathbf{x}$, we know that

$$|y| = \left| \lim_{n \to +\infty} f(\mathbf{x}_{k_n}) \right| = |f(\mathbf{x})| = |\infty|$$

Since $\mathrm{cset}(f, \{\mathbf{x}\})$ is a subset of $\overline{\mathbb{R}}_A$ by definition, y should be either $+\infty$ or $-\infty$. □

(g) $\forall \mathbf{x} \in \overline{D}_f \setminus D_f : \mathrm{cset}(f, \{\mathbf{x}\}) \subseteq \overline{\mathbb{R}}_A, \mathrm{vset}(f, \{\mathbf{x}\}) = \{\mathrm{NaN}\}$

Proof. By definition. □

(h) $\forall \mathbf{x} \notin \overline{D}_f : \mathrm{cset}(f, \{\mathbf{x}\}) = \emptyset, \mathrm{vset}(f, \{\mathbf{x}\}) = \{\mathrm{NaN}\}$

Proof. By definition. □

From the definition of vset-extension of f and from (a) through (h) it follows that for any $\mathbf{I} = (I_1, I_2 \cdots I_n) \in \mathbb{I}_r^n$

$$\mathbf{x} \in \mathbf{I} \Rightarrow f(\mathbf{x}) \in S(\text{vset}(f, \mathbf{I})) \subseteq S(f_V(\mathbf{I}))$$

since S, the semantic function introduced in *Definition* 9, is such that $S(\{\text{IaC}\}) = \mathbb{C}$. With the exception of (c) and (f), it also follows from the above that

$$\text{cset}(f, \mathbf{I}) \supseteq \text{vset}(f, \mathbf{I}) \cap \mathbb{R}_A$$

This concludes the proof of the theorem.

A.6. **Proof of theorem 3.6.**

Theorem (inclusion property)**.** *Let g be a function from $\overline{\mathbb{C}}^n$ to $\overline{\mathbb{C}}^m$ and let f be a function from $\overline{\mathbb{C}}^m$ to $\overline{\mathbb{C}}$. For any $X \subseteq (\overline{\mathbb{C}})^m$, we set $\tilde{f}(\mathbf{X}) = \{f(\mathbf{x}) \mid \mathbf{x} \in \mathbf{X}\}$. Then, for any $\mathbf{I} \in \mathbb{I}_s^n$, the following holds:*

$$\widetilde{f \circ g}(S(\mathbf{I})) \subseteq S(f_V(g_V(\mathbf{I})))$$

where S is the semantic function introduced in Definition 9.

Proof. From *Theorem 3.3* we know that for any $\mathbf{I} \in \mathbb{I}_r^n$,

$$\widetilde{g}(\mathbf{I}) \subseteq S(g_V(\mathbf{I})) \tag{9}$$

According to the definition of value set, we also have that for any $\mathbf{X} \subseteq (\mathbb{R} \cup \mathbb{S})^n$

$$\mathbf{x} \in S(\mathbf{X}) \Rightarrow g(\mathbf{x}) \in S(\text{vset}(g, \mathbf{X}))$$

and hence we obtain the following generalization of (9):

$$\widetilde{g}(S(\mathbf{I})) \subseteq S(g_V(\mathbf{I})) \qquad\qquad \mathbf{I} \in \mathbb{I}_s^n \tag{10}$$

It follows that

$$\widetilde{f}(\widetilde{g}(S(\mathbf{I}))) \subseteq \widetilde{f}(S(g_V(\mathbf{I})))$$

Applying (10) again, but now to $\widetilde{f}$, we find

$$\widetilde{f}(\widetilde{g}(S(\mathbf{I}))) \subseteq S(f_V(g_V(\mathbf{I})))$$

and hence

$$\widetilde{f \circ g}(S(\mathbf{I})) \subseteq S(f_V(g_V(\mathbf{I})))$$

□

Appendix B

In the paper we indicate that introducing, besides IaC, the special value IaC_0, turns out to be essential for sharpness in many instances. In this appendix we refine the definitions in *Section 3* to take into account the special value IaC_0. We recall from *Section* 3 that $\{\text{IaC}\}$ represents a set of values V where $V \subseteq \mathbb{C}$. Similarly, $\{\text{IaC}_0\}$ represents a set V where $V \subseteq \mathbb{C}_0$.

Definition B.1. *For any $X \subseteq \mathbb{R} \cup \mathbb{S} = \mathbb{R} \cup \{\infty, \text{IaC}, \text{IaC}_0, \text{NaN}\}$, we define the semantics $S(X)$ of X recursively as follows :*

$$\forall x \in \mathbb{R} \cup \{\infty, \text{NaN}\} : S(\{x\}) = \{x\}$$
$$S(\{\text{IaC}_0\}) = \mathbb{C}_0$$
$$S(\{\text{IaC}\}) = \mathbb{C}$$
$$\forall X \subseteq \mathbb{R} \cup \mathbb{S} : S(X) = \bigcup_{x \in X} S(\{x\})$$

Definition B.2. *Let f be a function of n variables and let $\mathbf{x} = (x_1, \dots, x_n)$. The value set of the function $f : \overline{\mathbb{C}}^n \to \overline{\mathbb{C}}$ with domain D_f evaluated at the point $\mathbf{x}$ is denoted by $\mathrm{vset}(f, \{\mathbf{x}\})$ and is defined as follows:*

$$\mathrm{vset}(f, \{x\}) = \begin{cases} \{f(x)\} & f(x) \in \overline{\mathbb{R}}_P \\ \{\mathrm{NaN}\} & x \notin D_f \\ \{\mathrm{IaC}_0\} & f(x) \in \mathbb{C}_0 \setminus \mathbb{R} \end{cases}$$

Definition B.3. *Let D_f be the domain of a function $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and let $s \in \{\mathrm{IaC}, \mathrm{IaC}_0\}$. Then we define*

$$\mathrm{vset}(f, \{s\}) = \begin{cases} \bigcup_{z \in S(\{s\})} \mathrm{vset}(f, \{z\}) & \forall z \in S(\{s\}) : f(z) \notin \mathbb{C} \setminus \mathbb{R} \\ \{\mathrm{IaC}\} \cup \bigcup_{z \in S(\{s\}) : f(z) \notin \mathbb{C}} \mathrm{vset}(f, \{z\}) & \exists z \in S(\{s\}) : f(z) \in \mathbb{C} \setminus \mathbb{R} \\ & \text{and} \\ & \exists z \in S(\{s\}) : f(z) = 0 \\ \{\mathrm{IaC}_0\} \cup \bigcup_{z \in S(\{s\}) : f(z) \notin \mathbb{C}} \mathrm{vset}(f, \{z\}) & \text{otherwise} \end{cases}$$

while, as before,

$$\mathrm{vset}(f, \{\mathrm{NaN}\}) = \{\mathrm{NaN}\}$$

Clearly, this last definition can be generalized in a straightforward way to functions of n variables.