Error estimation for the implementation of $\log_{10}$ in Arithmos

Johan Vervloet

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Theorem 1. *If a machine implementation for* $\log_{10} z$ *is based on this formula:*

$$\log_{10} z = \frac{\log z}{\log 5 + \log 2} \tag{1}$$

and the relative errors in the implementation of the logarithm, the addition and the division are bounded by $\frac{\epsilon}{12}$*, where* $\epsilon < 2\left(\sqrt{13} - 3\right)$*, then the relative error introduced by the implementation of* $\log_{10} z$ *is bounded by* ϵ*.*

Proof. The machine implementation of $\log_{10} z$ returns

$$f(z) = \frac{(\log z)(1+\epsilon_1)(1+\epsilon_5)}{((\log 2)(1+\epsilon_2) + (\log 5)(1+\epsilon_3))\,(1+\epsilon_4)} \tag{2}$$

where ϵ_1, ϵ_2 and ϵ_3 are relative errors caused by the implementation of log, ϵ_4 the relative error caused by the implementation of the addition, and $\epsilon 5$ the relative error caused by the implementation of the division.

Because

$$(\log 2)(1+\epsilon_2) + (\log 5)(1+\epsilon_3) = (\log 10)\left(1 + \frac{(\log 2)\epsilon_2 + (\log 5)\epsilon_3}{\log 10}\right) \tag{3}$$

it follows that

$$(\log 2)(1+\epsilon_2) + (\log 5)(1+\epsilon_3) = (\log 10)(1+\delta_1) \tag{4}$$

with

$$|\delta_1| \leq \max(|\epsilon_2|, |\epsilon_3|) \tag{5}$$

So we can rewrite expression (2) as

$$f(z) = \frac{(\log z)(1+\epsilon_1)(1+\epsilon_5)}{(\log 10)(1+\delta_1)(1+\epsilon_4)} \tag{6}$$

which is equivalent to

$$f(z) = \frac{(\log z)(1+\epsilon_1)(1+\epsilon_5)}{(\log 10)(1+\delta_2)} \tag{7}$$

where, assuming that $|\delta_1|, |\epsilon_4| < 1$,

$$|\delta_2| \leq 3\max(|\delta_1|, |\epsilon_4|) \tag{8}$$

Because

$$\frac{1}{1+\delta_2} = \sum_{k=0}^{\infty}(-\delta_2)^k = (1+\delta_3) \tag{9}$$

with

$$|\delta_3| \leq |\delta_2| \tag{10}$$

we can simplify expression (6) as

$$f(z) = (\log_{10} z)(1+\epsilon_1)(1+\epsilon_5)(1+\delta_3) \tag{11}$$

such that

$$f(z) = (\log_{10} z)(1+\delta_4) \tag{12}$$

where, assuming that $|\epsilon_1|, |\epsilon_5|, |\delta_3| \leq \frac{\sqrt{13}-3}{2}$,

$$|\delta_4| \leq 4\max(|\epsilon_1|, |\epsilon_5|, |\delta_3|) \tag{13}$$

If now $|\epsilon_1|, \ldots, |\epsilon_5| \leq \left|\frac{\bar{\epsilon}}{12}\right|$, then

$$|\delta_1| \leq \left|\frac{\bar{\epsilon}}{12}\right| \qquad \text{because of (5)} \tag{14}$$

$$|\delta_2| \leq \left|\frac{\bar{\epsilon}}{4}\right| \qquad \text{because of (8)} \tag{15}$$

$$|\delta_3| \leq \left|\frac{\bar{\epsilon}}{4}\right| \qquad \text{because of (10)} \tag{16}$$

$$|\delta_4| \leq |\bar{\epsilon}| \tag{17}$$

which, in combination with equation (12), concludes the proof. □